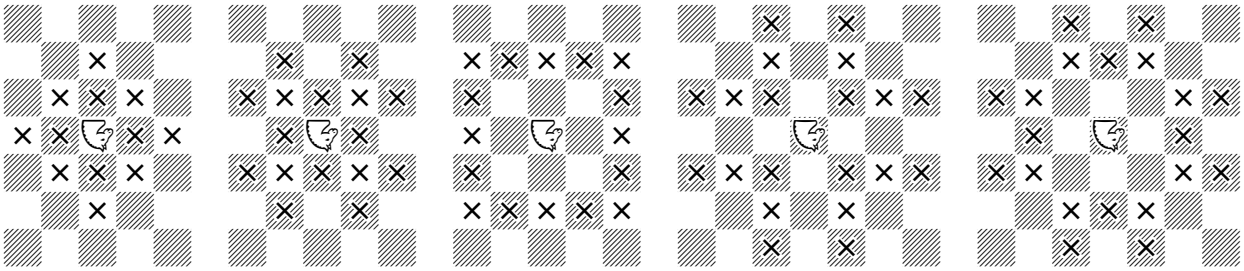
An apparently complete verdict on the ending with king and triplet leaper against king

John Beasley reporting work by Marc Bourzutschky, Noam Elkies, and himself, December 2014

In generalised chess, an *x,y leaper* is a piece which jumps *x* squares in one direction and *y* squares in the other (so the knight of ordinary chess is a 2,1 leaper). A *doublet leaper* is a composite of two separate *x,y* leapers (so the ordinary king, in so far as its properties of movement are concerned, is a 1,1/1,0 doublet), and a *triplet leaper* is a composite of three separate *x,y* leapers. In *Variant Chess* 47 (February 2005) I reported on work which appeared to give a complete picture of the ending of king and one doublet leaper against king and, and in *Variant Chess* 64 (August 2010) I reported what we had discovered thus far in respect of king and one triplet leaper. This document completes the picture.

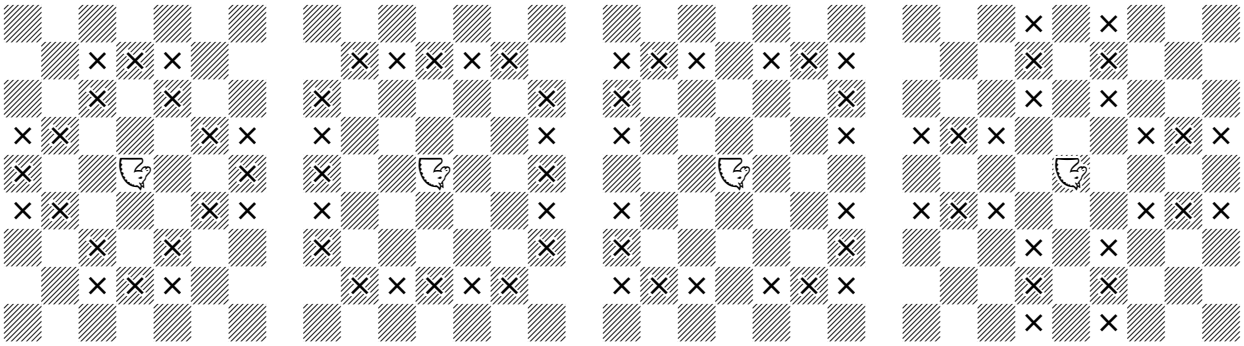
The situation as reported in *Variant Chess* 64

(a) In the five cases **2,0/1,1/1,0**, **2,1/1,1/1,0**, **2,2/2,1/2,0**, **3,1/2,1/1,1**, and **3,1/2,1/2,0**



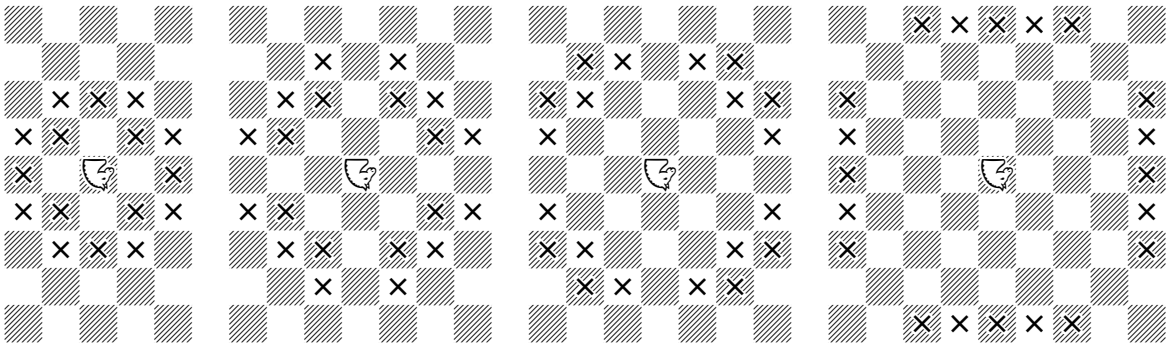
there were systematic procedures which forced a win however large the board might be.

(b) In the four cases **3,1/3,0/2,1**, **3,2/3,1/3,0**, **3,3/3,2/3,1**, and **4,1/3,1/2,1**



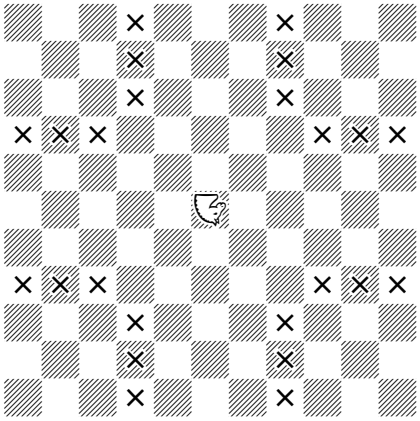
there were systematic procedures which appeared to force a win however large the board might be, but some gaps in the analysis remained to be filled.

(c) In the four cases **2,1/2,0/1,1**, **3,1/2,2/2,1**, **3,2/3,1/2,2**, and **4,2/4,1/4,0**



there seemed likely on the numerical evidence to be systematic procedures which would force a win however large the board might be, but no such procedure had been explicitly identified.

(d) In the case **5,2/4,2/3,2**



we were not in agreement.

(e) In all other cases, the lone king appeared to be able to hold out indefinitely provided that the board was sufficiently large.

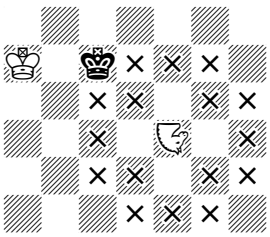
However, a run by Marc to investigate the behaviour of a 5,2/4,2/3,2 leaper on a 175x175 board, which was still in progress as *VC* 64 went to press, subsequently showed it to be a general win with greatest depth 10,657. This was entirely in line with what could have been predicted from the values for smaller boards, and strongly suggested that the anomalously high value for a 150x150 board reported in *VC* 64 was a casual glitch rather than the first intimation of a general rise. I reported this in a posting on www.jsbeasley.co.uk on 13 August 2012, in which I abandoned my previous position and aligned myself with Marc and Noam in believing that this ending would prove to be a general win however large the board might be

Subsequent developments

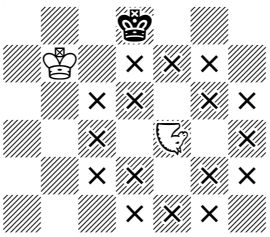
In the hope of finding systematic winning procedures for the five triplets for which no such procedure had not yet been identified (2,1/2,0/1,1, 3,1/2,2/2,1, 3,3/3,2/2,2, 4,2/4,1/4,0, 5,2/4,2/3,2), I asked Marc to send me specimen "best play" lines on boards up to 100x100, the Black king starting in the centre and the White men in the corners. He kindly did so, and I searched the play for configurations of the kings and leaper which repeated.

In most cases, such repetitions did indeed appear. Consider a 2,1/2,0/1,1 leaper, and suppose that we have position **1a** below with Black to move.

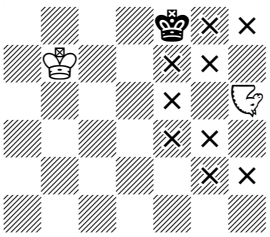
1a



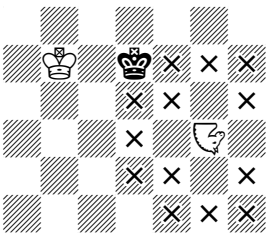
1b



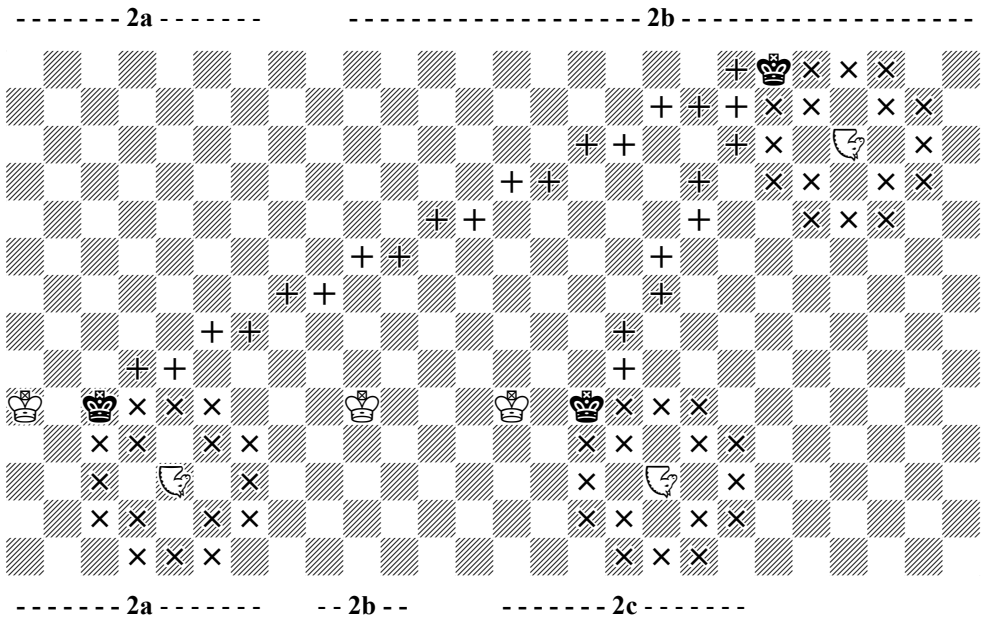
1c



1d



If we adopt the convention that the bottom left-hand square in a diagram is always "a1", the computer's best-play line, if we are trying to push the Black king towards the top edge but are still remote from all edges, is **1...Kd6 2 Kb5** (see **1b**) **Ke6 3 Lg4** (see **1c**) **Kd5 4 Lf3** (see **1d**), and although we have failed to gain any ground upwards we have pushed Black one square to the right. Assuming that Black can indeed do no better, we now have the kernel of a systematic winning procedure, since eventually the proximity of the right-hand edge will force Black to retreat. However, we need to satisfy ourselves (a) that Black is not able to outflank the White leaper by running to the right straight away (or, by dragging the leaper to the right to combat this, to create a gap between the king and leaper through which he can escape towards the bottom of the board), and (b) that if he tries to run across the front of the White king and escape to the left, which sooner or later he will have to do, he will have to retreat at least one rank upwards.



To examine the first of these possibilities, let us again consider position **1a**, repeated as **2a** above, and let us suppose that Black tries to escape by outflanking the White leaper to the right. Play again starts **1...Kd6 2 Kb5 Ke6 3 Lg4**, and if Black backtracks at once by **3...Kd5** we have **4 Lf3** as before and we have gained a file to the right. If instead Black plays **3...Kf7 4 Kc5 Kg7 5 Li5** and now backtracks, **5...Kf6**, we have **6 Kd5 Kf5 7 Lh3**, and we have gained three files to the right. If Black substitutes **5...Kh8 6 Kd5 Ki8 7 Lk6** and now backtracks, **7...Kh7**, we have **8 Ke5 Kh6 9 Lj4 Kg5 10 Li3**, and we have gained four files to the right. If he plays **7...Kj9 8 Ke5 Kk9 9 Lm7** and then backtracks, **9...Kj8**, we have **10 Kf5 Kj7 11 Ll5 Ki6 12 Kg5 Ki5 13 Lk3**, and we have gained six files to the right.

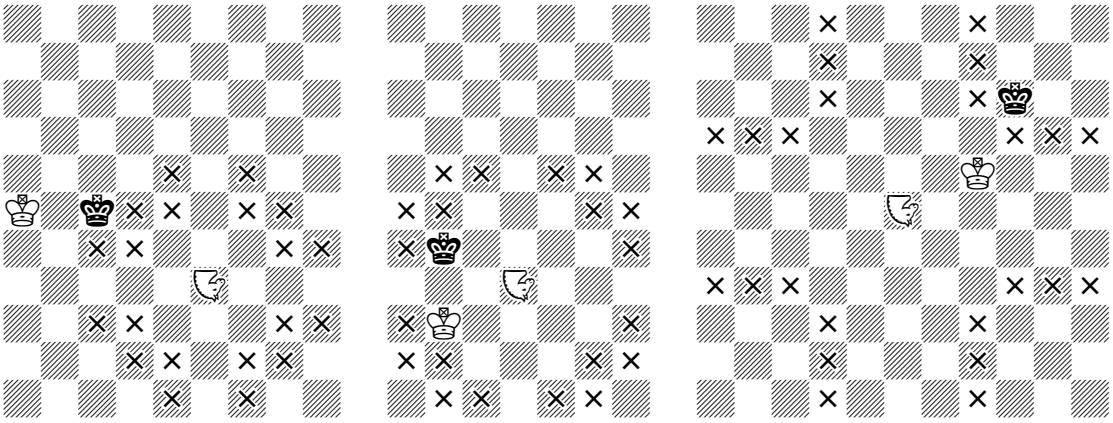
We now appear to have a pattern. If Black backtracks at move 4N–1, as after **17...Kt14 18 Kj5 Ku14 19 Lw12** (see **2b**, the right-hand file being file z), the backtrack sequence is typified by **19...Kt13 20 Kk5 Kt12 21 Lv10 Ks11 22 Kl5 Ks10 23 Lu8 Kr9 24 Km5 Kr8 25 Lt6 Kq7 26 Kn5 Kq6 27 Ls4 Kp5 28 Lr3** giving **2c**, and after 6N–3 moves we have gained 3N–2 files to the right; if he backtracks at move 4N+1, as after (from **2b**) a further **19...Kv15 20 Kk5 Kw15 21 Ly13**, the backtrack sequence is typified by **21...Kv14 22 Kl5 Kv13 23 Lx11 Ku12 24 Km5 Ku11 25 Lw9 Kt10 26 Kn5 Kt9 27 Lv7 Ks8 28 Ko5 Ks7 29 Lu5 Kr6 30 Kp5 Kr5 31 Lt3**, and after 6N moves we have gained 3N files to the right.

So running to the right is not going to help Black, and we see why the computer regards the immediate backtrack by 3...Kd5 as his best play.

It remains to be shown that if Black tries to run across the face of the White king, White can hold him back, if necessary by bringing his leaper round to the left, and can gain at least one rank upwards in the process. This is somewhat trickier and at one point I got stuck and had to ask one of Marc's programs to tell me the right move, but it can be done. In any case, it can fairly be argued that such a procedure *must* exist, else Black would be able to hold the draw by running across and attacking on each side alternately. A board 100x100 would surely be large enough to allow this.

This is a more relaxed version of the "one man and his dog" procedure which was described in *VC* 64. If we examine the moves closely, we see that however far the Black king may be from the White, White can always prevent him from crossing the SW-NE diagonal two below that through the White king. In the cases considered in *VC* 64, White could prevent the Black king, relative to the White, from crossing a line perpendicular to the line patrolled by the White king, and hence (allowing for the need to run across and control each side alternately) could pen him within a narrow and slowly advancing corridor based on the White king and perpendicular to the line patrolled by it. Here, the line behind which Black is penned relative to the White king is only at 45 degrees to the line which the White king is patrolling, so (again allowing for the need to control each side alternately) Black is merely penned within a quadrant relative to the White king, but even this is sufficient to ensure White's eventual success.

3 **4** **5**



Three of the four remaining triplets appear to yield to similar prodecures. Suppose that we have a 3,1/2,2/2,1 leaper in position **3** above, with Black to move, and that White is again trying to drive the Black king upwards; then the computer's best-play line continues **1...Kd7 2 Kb5 Ke8 3 Lh5 Kf8 4 Kc4 Kg9 5 Lj6 Kg8 6 Kd5 Kf7 7 Li4 Kf6 8 Kd6**, and although Black has resisted being pushed upwards we have **3** shifted three files to the right. It is of course necessary to prove, as we did with the 2,1/2,0/1,1 leaper, that Black cannot escape by trying to outflank the leaper on the right, and that if he tries to double back across the face of the White king White can hold him in and gain a rank upwards in the process, but although I have not proved these results I am satisfied that they are true. If they were not, the win on a 100x100 board would not exist.

Much the same is true of the 3,2/3,1/2,2 and 5,2/4,2/3,2 leapers, though here the direction of advance is diagonal. Suppose a 3,2/3,1/2,2 leaper in position **4**, with Black trying to avoid being driven NE; then the computer's best-play line continues **1...Kc5 2 Le7+ Kd5 3 Kc3 Kd6 4 Lb8+ Kc6 5 Kb4 Kb7 6 Lc5 Ka6 7 Ka4,** and although Black has resisted being driven NE we have position **4** shifted one square NW. The patterns with a 5,2/4,2/3,2 leaper are less clear-cut, but one that recurs involves position **5**. With Black trying to avoid being driven NE, the computer's best-play line continues **1...Kj9 2 Kh8 Kj10 3 Lb4 Ki10 4 Ld8+ Kh11 5 Kg8 Kg11 6 Kf9**, and although Black has resisted being driven NE we have position **5** shifted two squares NW. So in each case Black will eventually be constrained by the edge of the board, and will have to retreat..

There remains the 4,2/4,1/4,0 leaper. The win here is much quicker, and the best-play sequences are too short for any significant pattern to emerge. However, a few trials suggest that the White king and leaper can again gradually advance while keeping the Black king penned within a quadrant relative to the White, and the data recorded in *VC* 64 suggest that the ending can be won very much more quickly than in the cases considered above.

Summary

In some cases, the analysis is incomplete, but to fill in all the details would be tedious, it might well demand a better analyst than myself, and we have all moved on to other things. However, having examined the latest results supplied by Marc, I am now satisfied, as I think Marc and Noam have been for some time, that the complete list of triplet leapers which, with their king, can force mate from a general position against a bare king however large the board may be, is

2,0/1,1/1,0	2,1/2,0/1,1	3,1/2,1/1,1	3,1/2,2/2,1	3,2/3,1/2,2	3,3/3,2/3,1	4,2/4,1/4,0
2,1/1,1/1,0	2,2/2,1/2,0	3,1/2,1/2,0	3,1/3,0/2,1	3,2/3,1/3,0	4,1/3,1/2,1	5,2/4,2/3,2

and in all other cases the defending king can avoid mate if the board is sufficiently large.