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Three-man pawnless endings
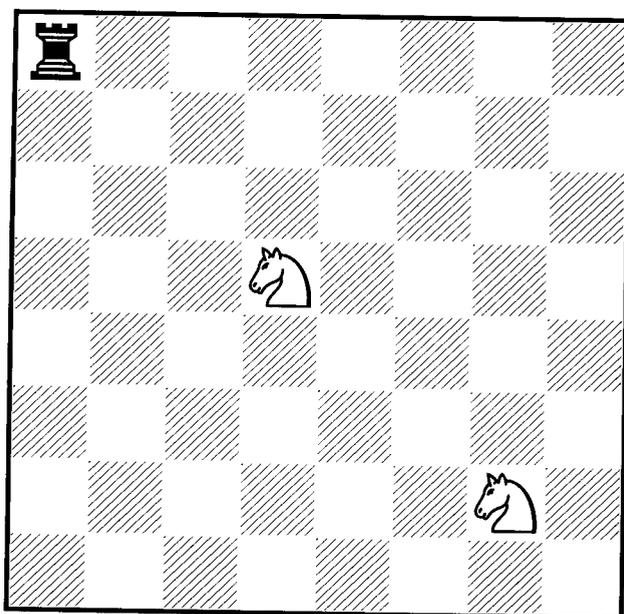
in

Losing Chess

Three-man pawnless endings in Losing Chess

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In January 1998, I produced an informal document entitled "Three-piece endings in Losing Chess" in which I examined all the three-man pawnless endings in Losing Chess with the aid of a computer-generated database. It was my intention at the time to wait for a year to see if anyone would claim to have done this work already (I was aware of the work by Gyorgy Evseev on two knights against one, quoted in the *British Chess Magazine* in November 1992 and subsequently described in an article "Finales de cavaliers en 'qui perd gagne' " by Evseev and Christian Poisson in *Rex Multiplex* in April 1993, but not of any other), and then to publish it more formally. However, nobody has made such a claim so far, and if anyone else had discovered the remarkable lose-a-move win with two knights against rook, where a player moving only a knight can contrive to transfer the move to his opponent, I think he would have done as I have done and published it everywhere in sight. I therefore presume, a little to my surprise, that this work had *not* been done previously by computer (though a number of endings had been examined by hand, notably by Fabrice Liardet in an article in *Schweizer Schach-Magazin* in August-September 1991), and the existence of a few unfortunate errors in the original document makes it desirable that a corrected version be issued.

This is therefore a revised version of the original document. I have retitled it to clarify its nature, and I hope I have removed all errors. The most serious of these was a mis-statement of a position of reciprocal zugzwang. To avoid double counting, the computer normalizes all positions, and its normalization is not always the most convenient for subsequent discussion. Some positions have therefore been renormalized, and one (in the ending R v K+B) was originally renormalized wrongly. I hope there is no such error in the present document, but as an insurance I have added appendices giving the database statistics and reciprocal zugzwangs as actually generated by the computer. Everything displayed in the fixed-pitch "Courier" font has been copied from computer output with no change beyond formatting into columns, but positions in the proportionally-spaced font used for the bulk of the text may have been renormalized and should be checked against Appendix 2 if there is any doubt.

I have no authority to waive the rights of others, but in so far as anything in this document is original with me I am happy that it be freely copied or quoted. I ask only that there be appropriate acknowledgement. If it appears that any of this work has been performed previously, other than as acknowledged in the text, or if any error exists in the present document, I shall be glad if the matter is brought to my notice.

My thanks to Fabrice Liardet for reporting the errors in the original document, and for other valuable comments. Since the original document was written, Laurent Bartholdi has generated a database covering all three-man positions with and without pawns, but I have not extended the present document beyond pawnless positions.

Preliminary note 1: "trivial" endings

Many endings in Losing Chess are "trivial": the player to move is forced to make an immediate capture, or can win by an immediate sacrifice. These trivial endings distort raw computer statistics to no useful purpose, and we shall always disregard them. The one-against-one draw with bishops of opposite colours is also trivial.

In the original document, an ending was also normally disregarded if a player could force his opponent to make a disadvantageous capture on his first or second move (a so-called "short" ending), but I now consider this a mistake and in the present document a short ending is treated like any other non-trivial ending.

Preliminary note 2: two-man pawnless endings

Although these are well known, it is convenient to list them here, partly for completeness and partly because they provide a gentle introduction to some ideas that we shall use later.

Endings in which each side has a line-moving piece (queen, rook, or bishop) are normally trivial, but there are exceptions in which a player can attack his opponent without allowing him to sacrifice in return. A queen or rook can do this against a bishop (for example, Q/Rb1 against Bb8) and a bishop against a rook (Bb1 against Rd3), but a queen cannot do it against a rook. All these are wins for the "attacking" side, because his opponent must move away and the attacker then sacrifices on the square which he has just been attacking.

Queen, rook, or bishop against knight are normally wins, but there are a couple of wins by "attack and wait" for knight against rook (typically, Nb1 against Rd2) and one for knight against bishop (Na2 against Bc1). Here and elsewhere, we count positions which can be rotated or reflected into each other as the same. There is also a win by "domination" for knight against bishop (Ne6 against Be1). The knight does not threaten anything in this position, but the bishop has no safe move and will have to allow the knight to sacrifice itself next move.

Knight against knight is a win for whoever is to move when the knights are on squares of the same colour. There are no exceptions, not even trivial ones.

Queen or rook against king are wins. **Bishop against king** is normally a draw, but there are three positions in which the bishop wins by domination (Ba4 against Ka1, Bc4 against Kc1, Bd4 against Kd1). Set Bb4 against Kb1, however, and the king can hold the draw by playing Ka1.

King against knight is normally a win, but there are some "attack and wait" wins for the knight against a king on the edge (typically, Nb2 against Kd1) and one "domination" win against a king in the corner (Nd4 against Ka1).

King against king is a draw.

The rest of this document will consider only the normal 8 x 8 board, but we may mention in passing that there are a few cases in which **the result may depend on the board size**. Qg3 to move against Kc1 wins only by playing to g8, h3, or h4, and on a 7 x 7 board this position is a win for the king. Be1 to move against Ne6 loses on 8 x 8 and 9 x 9, but wins on 10 x 10 and beyond. King against knight is a win on all boards up to and including 12 x 12, but the knight may be able to run indefinitely on a 13 x 13 or any larger board.

Preliminary note 3: castling

There is no three-man position in Losing Chess in which castling is the only move to win or to draw. We can therefore ignore castling altogether.

Preliminary note 4: move counts in computer databases

It is a property of computer databases that they normally count moves *up to a decisive change in material*, not to the final move of the game. Set wKe4 against bRa8 and bKc6, and the computer says that Black can win in one move by playing ...Kd5 since the capture KxK gives a lost position with K v R. In fact a White player who has nothing better to do with his time can drag out the lost K v R ending for a further 17 moves, but such behaviour is normally ignored. It is possible to construct databases so that they count moves up to the final move of the game and there are situations where this is desirable, but in the present circumstances it would tend to obscure what is going on rather than to clarify it.

We now proceed to three-piece endings. We always assume that White always has the single piece, and except where we are considering positions of reciprocal zugzwang we normally assume that it is the single piece to move.

1. Lone queen

Queen against queen and anything and **queen against rook and anything** are always trivial with the queen to move, as is **queen against two bishops**.

Queen against bishop and knight offers three non-trivial wins for the queen to move (Qg4 or Qh3 against Ba8/Nb7, Qh3 against Ba8/Nc6) and one non-trivial loss (Bb8/Nc7 against Qh4). In the latter position, whoever is to move loses.

Queen against two knights offers 56 non-trivial wins for the queen to move and 21 non-trivial losses. In two cases, the knights can hold out until move 3 (Qe1 against Na8/Nb8 or Nh7/Nb6). The latter case illustrates the computer's method of counting, since it gives the optimal line of play as 1 Qe2 Na8; it regards 1...Nc4 as inferior because 2 QxN immediately gives a won two-man ending for the queen, even though the final move of the game (if Black plays it out to the bitter end) occurs no sooner.

There are 21 non-trivial positions in which the side to move loses, listed by the computer as follows:

Na8/Ne8 v Qh3	Nc8/Ne6 v Qh1	Nc7/Nd6 v Qg1
Na8/Nc7 v Qh4	Nc8/Nc4 v Qh1	Nc7/Nb4 v Qg1
	Nc8/Nc2 v Qh5	
Nb8/Nc8 v Qh5		Nd7/Ne6 v Qh1
Nb8/Nd6 v Qg1	Nd8/Ne7 v Qh2	Nd7/Nf5 v Qa2
Nb8/Nb4 v Qg1	Nd8/Nf6 v Qa3	Nd7/Nc4 v Qh1
Nb8/Nb2 v Qg5	Nd8/Nc5 v Qh2	
Nb8/Nc1 v Qh4	Nd8/Nf4 v Qa1	Nd6/Nf4 v Qa1
	Nd8/Nh4 v Qa1	

Fabrice Liardet points out that these are precisely the 21 non-trivial losses for the queen to move, and that they all have the property that the queen is immediately dominated.

Queen against king and bishop introduces our first drawn positions. There are two such, Qh4 against Kc6 and Ba8 or Bb7, and Paul Byway exploited them in a study published in 1995. Suppose bBa8. White's only safe move is 1 Qe1; Black's only safe reply is 1...Bb7, giving the other drawn position; White can only play 2 Qh4;

Black must reply 2...Ba8, and we are back where we started.

There are nine non-trivial wins for the queen to move, and nine non-trivial losses. In two of the latter, Qf2 against Bc8/Kd7 and Qg2 against Bd8/Ke7, the queen can hold out until move 3 (taking Qf2 against Bc8/Kd7, 1 Qg1 Ke6 etc).

There are three non-trivial positions in which the side to move loses: Bc8/Kd7 against Qf2, Bd8/Ke7 against Qg2, Be8/Kf7 against Qh2. Why not Bb8/Kc7 against Qe2? Because White to play can win by Qe1 or Qh5.

Queen against king and knight is a tricky ending. The computer counts 127 non-trivial wins for the queen to move, 172 non-trivial losses, and 19 draws; many positions require detailed analysis, but we may note that there are no non-trivial positions with the knight in the four central squares and that all non-trivial positions with the knight on one of the twelve squares surrounding the centre are lost for the queen. (We shall see in many endings that the player with a knight should normally get it as near to the centre as he safely can.) The longest win occurs with Qg5 against Na8/Kb8: 1 Qh5 Kb7 (best) 2 Qd1 Kb6 3 Qg4 Ka6 4 Qe6/Qg6 (a remarkable resource) and Black has an unenviable choice between moving his king to a defended square (4...Kb6), defending it (4...Nc7), shielding it by a defended knight (4...Nb6), or moving it away and allowing White to sacrifice on the square it has just left.

The longest loss occurs with Qa1 against Nf8/Kg4, when the computer plays

1	Qa1-b2	!	Kg4-h4	!	5	Qa1-b2	!	Kh4-g4	!
2	Qb2-a1	!	Kh4-g3	!	6	Qb2-a1	!	Kg4-f3	!
3	Qa1-c1	!	Kg3-h3	!	7	Qa1-a5	!	Kf3-g2	!
4	Qc1-a1	!	Kh3-h4	!					

with a sacrifice after White's next move. Here and elsewhere in computer output, " ! " indicates a unique optimal move: other moves by the losing side lose more quickly, other moves by the winning side either delay the win or forfeit it altogether. The play from Black's 1st move to his 5th is a "lose-a-move" manoeuvre which returns to the same position with White to play.

The computer lists the following positions as drawn:

```
Na8/Kg8  v  Qd1                    Nb8/Ka4  v  Qf2
Na8/Kf7  v  Qd2                    Nb8/Ka3  v  Qg5
Na8/Kg7  v  Qe2,d1,e1              Nb8/Kb3  v  Qh5,g1
Na8/Kh7  v  Qe2,d1,e1,f1
Na8/Kf6  v  Qd1                    Nc8/Kh7  v  Qf3
Na8/Kg6  v  Qe1                    Nc8/Kh6  v  Qf1
Na8/Kh6  v  Qe1,f1
```

There are 13 non-trivial positions in which the side to move loses, listed by the computer as follows:

```
Nb8/Kd8  v  Qf2        Nd8/Kc8  v  Qa3        Nb7/Kf8  v  Qh3
Nb8/Kb7  v  Qg5        Nd8/Ke8  v  Qg3
                       Nd8/Kf8  v  Qh2        Nc7/Ka4  v  Qf2
Nc8/Ke8  v  Qg2        Nd8/Kb1  v  Qg3
Nc8/Ka5  v  Qf3        Nd8/Kf1  v  Qa3        Nc6/Kf8  v  Qh3
Nc8/Ka3  v  Qf1
```

In addition, there are two positions in which the queen loses with the move but can draw without it (Nb8/Kc3 against Qh5, Nc8/Kh5 against Qf1), and five in which the same is true of the pieces (Na8/Kf7 against Qd2, Nb8/Ka4 against Qf2, Nb8/Ka3 against Qg5, Nb8/Kb3 against Qg1, Nc8/Kh7 against Qf3).

Some positions in this ending were exploited in 1995 by Paul Byway and in 1997 by Vincent Geerlings.

Queen against one king is a win for the queen, but the queen has limited freedom of action. The addition of a second king restricts her further, and **queen against two kings** is not easy to summarize. The computer counts 63 non-trivial wins for the queen to move, 121 non-trivial losses, and 105 draws. The list of draws is as follows:

```
Ka8/Kc8  v  Qh6,h5                 Ka8/Kg6  v  Qe1
Ka8/Ke8  v  Qc3,c2,c1              Ka8/Kh6  v  Qc4,e1
Ka8/Kf8  v  Qd3,h3,c2,d2,c1,d1     Ka8/Kh2  v  Qf6,c4
Ka8/Kg8  v  Qd2,e2,c1,d1,e1        Ka8/Kh1  v  Qe6,f6
Ka8/Kh8  v  Qd2,c1,d1
Ka8/Kb7  v  Qg5,h5,h4              Kb8/Kc8  v  Qg6
Ka8/Ke7  v  Qc2,c1                 Kb8/Kf8  v  Qd3,d2,d1
Ka8/Kf7  v  Qd2,c1,d1              Kb8/Kg8  v  Qd2,d1
Ka8/Kg7  v  Qe2,d1,e1              Kb8/Kb7  v  Qd2
Ka8/Kh7  v  Qe2,d1,e1,f1          Kb8/Kf7  v  Qd2,d1
Ka8/Ke6  v  Qc1                    Kb8/Kg7  v  Qe2,d1,e1
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```
Kb8/Kh7 v Qe2,d1,e1,f1              Kc8/Kg6 v Qe1
Kb8/Kf6 v Qd1                       Kc8/Kh6 v Qa2,f2,e1,f1
Kb8/Kg6 v Qe1                       Kc8/Kb3 v Qh6,h5,g1
Kb8/Kh6 v Qe1,f1                    Kc8/Kb2 v Qh5
Kb8/Ka4 v Qg6,h6
Kb8/Ka3 v Qg6,h6,g5,h5,f1          Kd8/Ke8 v Qb2,a1,b1
Kb8/Kb3 v Qh6,g5,h5                Kd8/Kc7 v Qa1
Kb8/Kc3 v Qh5                      Kd8/Ke7 v Qg1,h1
Kb8/Kb2 v Qh5,h4
Kb8/Kc2 v Qh4                      Kb7/Kf7 v Qd2,d1
Kb8/Kb1 v Qh5                      Kb7/Kg7 v Qd1
Kb8/Kc1 v Qh4                      Kb7/Kf6 v Qd1
Kb8/Kd1 v Qf6                      Kb7/Kg6 v Qe1

Kc8/Kd8 v Qa2,f1,g1                Kc7/Kg6 v Qe1
Kc8/Kg7 v Qe2,e1
```

Most are similar in character to the draws with queen against king and knight, the queen having only one or two safe moves at each stage, but the positions with adjacent or near-adjacent kings introduce a new motif. Consider Ka8/Kb7 against Qg5, which is one of the positions where the queen has the initiative. The queen would defeat either king on its own by staying just out of range and waiting for its opponent to retreat, but with two kings Black can set one forward as a bastion and play tempo moves with the other. So Black keeps his forward king at b7 and moves his other king carefully around the squares a8-b8-a7 (one square is always safe), and White can never approach.

The longest win for the kings occurs with Ka7/Ke6 against Qc1, when the computer plays

```
1  Qc1-c2 !    Ke6-e7 !         4  Qc2-c1       Ka6-a5 !
2  Qc2-c1 !    Ka7-a6 !         5  Qc1-h1 !     Ke8-f7 !
3  Qc1-c2 !    Ke7-e8 !         6  Qh1-c1       Kf7-e6 !
```

with a sacrifice after White's next move. White can also try 4 Qh2, but the sequel 4...Ka5 5 Qg2 Kb4 6 Qh2 Kc4 is no better.

The only long wins for the queen occur with Qh6 or Qh5 against Kb8/Kc8. White plays 1 Qg6; Black must reply 1...Ka8 to avoid immediate loss; and White plays 2 Qb1 with a sacrifice next move.

There are four non-trivial positions in which the side to move loses (Kc8/Kd8 against Qh6, Kc8/Ke1 against Qg6 or Qa3, Kd8/Ka5 against Qf3). There are also six positions in which the queen loses with the move but can draw without it (Ka8/Kd8 against Qh6, Kb8/Ka4 against Qf6, Kb8/Kh2 against Qf6, Kc8/Kd8 against Qa1, Kd8/Kc7 against Qa2, Kd8/Ke7 against Qg2) and three in which the same is true of the kings (Ka8/Kh6 against Qc4, Kb8/Kc8 against Qg6, Kb8/Kd1 against Qf6).

2. Lone rook

Rook against queen and anything and **rook against rook and anything** are always trivial with the rook to move.

Rook against two bishops, with the rook to move, is always trivial with unlike bishops, but with like bishops it offers 32 non-trivial wins for the rook and 8 non-trivial losses. These are typified by Rd5 against Ba8/Bb7 (the rook wins) and Rd5 against Ba8/Bg8 (it loses). With the bishops to move, Fabrice Liardet points out some short but interesting non-trivial losses, for example Ba5/Bd1 against Rh8. Here, a bishop must sacrifice itself immediately if Black is to stay alive, but the only two sacrifices (...Bd8 and ...Bh5) each lead to exceptional positions where B v R loses even though the bishop has the move.

Rook against bishop and knight offers 321 non-trivial wins for the rook to move, 14 losses, and 24 draws. Twelve of the draws are given by bNd6, bB on b8 or c7, and wR somewhere in the rectangle g3-h3-h1-g1, where it is easily seen that if bB moves off the line b8-c7 or wR moves outside the rectangle g3-h1 the other side can sacrifice immediately. Ten of the other positions are similar (Rh2 or Rh1 against Bb8/Ne5, the same against Bd6/Ne5, Rg2/h2h1 against Bb7/Nd5, and the same against Bc6/Nd5). In the other two positions, we have Rh2 against Bb8/Nc5 or Bc7/Nc5. Here White plays 1 Rh1; if Black sacrifices by 1...Bh2, he gives himself a lost N v R ending, so he must hide his bishop by 1...Ba7 or 1...Bb6; White plays 2 Rg1, and we have a reflection of one of the positions already described. The positions with Rh1/h2 against Bb8/d6 and Ne5 were exploited in a study published by R. Sekhar and R. Shankar in 1987. Note that Black cannot play ...Bc7 in these positions because White can reply Ra2; the sacrifice ...Ba5 gives White a winning ending with R v N.

The non-trivial wins for the rook also offer points of interest. The longest has Rh3 against Bc8/Nc6, and White starts 1 Rh1/Rh2. Black will lose if he sacrifices on h3, so he hides his bishop by 1...Bb7. White now plays

2 Rg1/Rg2, and Black must play 2...Ba8 to keep his bishop hidden. But this restricts the bishop to the long diagonal, and now White can attack the knight, 3 Rg6 say, and Black cannot leave it without defence.

The non-trivial losses for the rook are typified by Rh1 against Ba8/Ne5. The rook must move away, and while a sacrifice on h1 would lose Black always has a winning alternative sacrifice. There is also a loss by domination (Rh1 against Bb7/Ne4). Note however that the apparently similar position with Rh1 against Ba8/Ne4 is a win for the rook. White can play 1 Rh7/Rb1, and the only sacrifice available to Black is a losing one; alternatively, he can attack the knight, and Black cannot leave it without defence.

Rook against two knights is one of the most interesting endings. The computer counts 641 non-trivial wins for the rook to move, 22 non-trivial losses, and 197 draws.

To get a feel for the material, let us set the rook at h1, one of the knights in an oppressive position at e4, and the other knight in the top left-hand corner.

- With bN on a8, White soon wins. He plays 1 Rh2, and Black must retreat since 1...Nd2/Nf2 2 RxN would be a win for the rook against the remaining knight. Black's two sensible moves (1...Nc5 and 1...Nd6) are of equal value; let us choose 1...Nd6. White plays 2 Re2, and bN must retreat again because even 2...Ne8 3 RxN would give the win to White. Black's best move is 2...Nb5; White plays 3 Re4 forcing Black to retreat for the third time; and we have 3...Na7 4 Rd4 with a sacrifice next move.

- With bN on b8, the game is drawn. White can play 1 Rg1 forcing Black to retreat, but after 1...Nc5 he has nothing better than 2 Rf1/Rg2. Black can now hold the draw most simply by keeping one knight at c5 and playing the other back and forth between c6 and b8. This restricts wR to the rectangle f2-h2-h1-f1, and progess is clearly impossible.

- With bN on b7, wR is dominated and any move is immediately fatal.

If it is Black to move in this last position, he must lift the domination, but he can still win. The computer gives the following:

1	......	Nb7-c5	6	Rh8-h7 !	Nc3-b1 !
2	Rh1-h8 !	Nc5-b3 !	7	Rh7-h8 !	Nb1-d2 !
3	Rh8-h7 !	Nb3-d4 !	8	Rh8-h7 !	Nd2-b3 !
4	Rh7-h1 !	Nd4-b5 !	9	Rh7-h8 !	Nb3-c5 !
5	Rh1-h8 !	Nb5-c3 !	10	Rh8-h1 !	Nc5-b7 !

Apart from the opening move, where 1...Nd6 is equivalent by symmetry, each of these is a unique optimal move, and at the end we have returned to the original position with White to play. We can therefore add this to the small and delightful class of chess positions in which a player moving only a knight can contrive to transfer the move to his opponent.

The play rewards detailed study. At move 1, moves by the e-knight can be shown to lose, and 1...Nd8/Na5 allow White to hold the draw (after say 1...Nd8, White can play 2 Rg1, driving bNe4 from its commanding position). Play is now straightforward until after Black's third move, when we have the knights side by side on d4 and e4 and the rook on h7. If White now plays 4 Ra7, we have 4...Ng3 5 Ra8 Ngf5 6 Ra1 Ng7 echoing the original domination. If he plays 4 Rh8, we have 4...Nc3 5 Rh7 (5 Rg8 is equivalent by symmetry) Nce2 6 Ra7 (if 6 Rh8 then 6...Nb5 dominates wR at once) Ng3 and we have the same position as after 4 Ra7 Ng3. This leaves his actual move 4 Rh1, but even this allows Black to dictate the play (White's only significant choice occurs at move 8, when he can play Ra8 instead of Rh7, but then 8...Ng5 dominates wR at once). It may seem strange that Black has to play his knight right down to b1 at move 6, but nothing else wins. If at any time he puts his knights an even number of squares apart on the same rank or file, White can move to the line of bisection and win at once (for example, 4...Ne6? 5 Rh5), and this seriously limits his freedom of action.

The position with the knights side by side on d4 and e4 is thus seen to be a key winning position in this ending. The rook has only three pairs of temporarily safe squares, h8/a8, h7/a7, and h1/a1, and each loses in due course.

A complete list of non-trivial positions in which the rook to play loses is as follows:

Nb8/Ne5, Rh2 or h1;	Nd7/Ne5, Ra2 or h1;	Nd6/Ne4, Ra1 or h1;
	Nd7/Nc4, Rg1;	
Nb7/Ne4, Rh1;	Nd7/Ne4, Ra1 or h1;	Nd5/Ne5, Rh8, h2, or h1;
	Nd7/Ng4, Ra1;	Nd5/Ne4, Rh1.
Nc7/Ne6, Rh3;		
Nc7/Nd4, Rg1 or h1;	Nc6/Nd5, Rg2 or h1;	
	Nc6/Ne5, Rh2 or h1;	

We have already met several of these during the play above; in the case of the rest, Black can either play to a

position with his knights side by side in the centre and then reel off the win already shown, or play to join this solution after a few moves. With Nd5/Ne4 against Rh1, for example, we have 1 Rh8 (1 Ra1 is equivalent) Ndc3 2 Rh7 Nb1, and we are back on familiar territory.

If Black cannot establish a knight in the centre, it is White who has the winning chances, and the defensive strategy is to place one knight as a bastion as far from the corner as possible and play tempo moves with the other. The computer gives the complete list of draws as follows:

```
Nb8/Ne6  v  Rh2,h1               Nc7/Nb6  v  Rh1
Nb8/Nb5  v  Re2,f2,g2,h2,e1,f1,  Nc7/Nd6  v  Rg3,h3,g2,h2,g1,h1
            g1,h1                Nc7/Nb5  v  Rf2,g2,h2,f1,g1,h1
Nb8/Nc5  v  Rf2,g2,h2,f1,g1,h1   Nc7/Nd5  v  Rg2,h2,g1,h1
Nb8/Ne4  v  Rh1                  Nc7/Nb4  v  Rf1,g1,h1
                                 Nc7/Nc4  v  Rf1,g1,h1
Nc8/Nd7  v  Rg4,h4,g3,h3,g2,h2,  Nc7/Ne4  v  Rh1
            g1,h1
Nc8/Nd6  v  Rg3,h3,g2,h2,g1,h1   Nd7/Ne7  v  Ra4,a3,a2,a1
Nc8/Nb5  v  Rf2,g1,h1            Nd7/Nc6  v  Rg3,h3,g2,h2,g1,h1
Nc8/Nc5  v  Rf2,g2,h2,f1,g1,h1   Nd7/Nd6  v  Ra3,g3,h3,a2,g2,h2,
Nc8/Nd5  v  Rg2,h2,g1,h1                     a1,g1,h1
Nc8/Ne5  v  Rh2,h1               Nd7/Ne6  v  Ra3,h3,a2,h2,a1,h1
                                 Nd7/Nf6  v  Ra3,a2,a1
Nd8/Ne7  v  Ra4,h3,h2,h1         Nd7/Nb5  v  Rg2,h2,h1
Nd8/Nc6  v  Rg3,h2,h1            Nd7/Nc5  v  Rg2,h2,g1,h1
Nd8/Ne6  v  Ra3,h3,a2,h2,a1,h1   Nd7/Ne5  v  Rh2,a1
Nd8/Na5  v  Rg2,h1               Nd7/Nf5  v  Ra2,a1
Nd8/Nc5  v  Rg2,h2,g1,h1         Nd7/Ng5  v  Ra2,a1
Nd8/Nd5  v  Ra2,g2,h2,a1,g1,h1   Nd7/Nc4  v  Rh1
Nd8/Ne5  v  Ra2,h2,a1,h1         Nd7/Nd4  v  Ra1,g1,h1
Nd8/Nf5  v  Ra2,a1               Nd7/Nf4  v  Ra1
Nd8/Ng5  v  Ra2,a1
Nd8/Nc4  v  Rg1,h1               Nc6/Nd6  v  Rg3,h3,g2,h2,g1,h1
Nd8/Ne4  v  Ra1,h1               Nc6/Nd5  v  Rh2
Nd8/Ng4  v  Ra1                  Nc6/Ne4  v  Rh1
Nd8/Nh4  v  Ra1                  Nd6/Ne6  v  Ra3,a2,a1
                                 Nd6/Nc5  v  Rg2,h2,h1
Nb7/Ne7  v  Rh4                  Nd6/Nd5  v  Ra2,g2,h2,a1,g1,h1
Nb7/Nc6  v  Rf3,g2,h2,h1         Nd6/Ne5  v  Ra2,h2,a1,h1
Nb7/Ne6  v  Rh3,h2,h1            Nd6/Nf5  v  Ra2,a1
Nb7/Nd5  v  Rh1                  Nd6/Nf4  v  Ra1
Nb7/Ne5  v  Rh1
                                 Nd5/Ne4  v  Rh8
Nc7/Nd7  v  Rg4,h3,h2,h1
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The key positions are those in which the knights are side by side, and we see that d7/e7, d7/d6, c6/d6, d6/e6, and d6/d5 draw in all cases, and that c7/d7 draws against g4/h3/h2/h1 but not otherwise. In the last case, we might also expect draws with wR on h4 or g3-g1, but White can play 1 Rg4 and we have 1...Na8 (if Black gives ground by 1...Nb8 then 2 Rf4 wins easily) 2 Re4 Nb8 (now Black must give ground whether he likes it or not, because 2...Ne5 3 RxN is a win for the rook) 3 Re5. With wR on g4 or h1-h3, however, White has nothing better than 1 Rh4, and now Black can play 1...Ne8. White must play 2 Ra4, else 2...Nd6 will draw; Black plays 2...Ng7; White must retreat, say by 3 Ra3, and Black has gained enough ground to draw (3...Ne6 etc).

Two other positions are of interest: Nc6/Nd5 against Rh2, and Nd5/Ne4 against Rh8. Here the knights have gained a strong position, and it is the rook which is trying to hold the game. In the first case, Black threatens 1...Ne5 and 1...Nd4, in each case with the standard win, but White can hold him at bay by 1 Rg2 and now it is Black who has to be careful (only 1...Ne8 and the symmetrically equivalent 1...Na5 hold the draw). In the latter case, White would lose if his rook were on a8, as we have already seen, but with the rook on h8 White can play 1 Rh1 (or 1 Ra8) and Black can never achieve a winning position.

There is a single non-trivial position in which the side to play loses (Nc7/Ne6 against Rh3), but there are five positions in which the rook loses with the move but can draw without it (Nd7/Ne5 against Ra2, Nd7/Nc4 against Rg1, Nd7/Ng4 against Ra1, Nc6/Nd5 against Rg2, Nd5/Ne4 against Ra8) and seven in which the same is true of the pieces (Nc8/Nb5 against Rf2, Nd8/Ne7 against Ra4, Nd8/Nc6 against Rg3, Nd8/Na5 against Rg2, Nb7/Ne7 against Rh4, Nb7/Nc6 against Rf3, Nc7/Nd7 against Rg4).

It is a finely poised ending, apparently first examined by Fabrice Liardet in 1991.

Rook against king and bishop, with the rook to move, is normally won for the rook except when trivially lost,

but the computer counts five non-trivial losses and 112 draws. The non-trivial losses are given by Re4 against Ba8/Kg6 and Rh1 against Ba8/Kf6, Bd5/Kb7, Bd5/Kf7, or Bd5/Kf3, the rook having no good move in any case and Black being able to sacrifice both his men. However, the position of bK is crucial. White to play loses with Rh1 against Bd5/Kf7, but if bK is on f8 he wins by Rc1 and if bK is on f6 he wins by Ra1.

The draws are all characterized by bK shielding bB, either at the start (Ba8/Kc6 against Rf4) or after the first move. The latter case is typified by Ba8/Kc7 against Re4, where bB is attacking wR. The wR moves away (say 1 Rf4) and Black cannot sacrifice bB without conceding a losing K v R ending; so he shields his bishop, and we have a position of the first kind.

Some of the wins with the rook are also interesting, and one of the longest has Rh3 against Ke5/Bd7. White starts 1 Rh2 (1 Rh1 only draws), and 1...Bh3 concedes a loss with K v R. Black therefore shields his bishop, and 1...Kf5 is his better move. Now White plays 2 Rh1, and Black has a dolorous choice: 2...Bc6 3 Rh7 (only move to win), or 2...Ke6 3 Ra1 (best, though some other moves win more slowly), or 2...Bc8/Be6 3 Ra1 (this time White has alternatives just as good). In each case, White's next move will either sacrifice wR or capture bB with a winning R v K ending.

There is one non-trivial position in which whoever is to move loses (Kf3/Bd5 against Rh1) and one in which the pieces lose with the move but can draw without it (Kf4/Bd6 against Rh1). This latter position was misreported in the original document.

Rook against king and knight offers no non-trivial losses for the rook to move, but there are 1677 draws. To hold the game, the pieces must come close together, and a non-trivial position is always drawn if the king and knight are within two squares of each other and the king is away from the edge. However, there are some opportunist wins for the rook if the king is on the edge, even if the knight is close. A longest such win has Rg3 against Ke8/Nc7, with play

1	Rg3-b3 !	Nc7-e6 !	3 Rb5-g5 ! Nf8-h7 !
2	Rb3-b5 !	Ne6-f8 !	4 Rg5-g4

and a sacrifice to follow.

If the rook can safely command a line between the pieces, it wins. To get a feel for the material, let us suppose the knight to be on b6, the rook on e1, and the king somewhere in the rectangle g8-h8-h3-g3. If the king is on the eighth or seventh rank, 1 Re5 forces the knight to move to the left; if the king is on the sixth rank, 1 Rc1 forces the knight to the left; if the king is on the fifth or fourth rank, 1 Re7 forces the knight to the left; if the king is on the third rank, 1 Rb1 wins at once. The rook is a more precise attacker than the queen, and there are no draws against widely separated defenders such as occurred with a queen against the same men. The only cases where the rook cannot immediately force the knight to move further away from the king arise when it is on the eighth rank and the king is on the seventh or sixth, and in these positions the rook transfers its attention to the king and gradually cramps it as in the two-man ending R v K. A longest win occurs with Rd5 against Na8/Kf7, with play

1	Rd5-d4	Kf7-f6 !	6	Rf4-f8 ! Nc7-b5 !
2	Rd4-d3	Kf6-g7 !	7	Rf8-e8 ! Kh6-g6 !
3	Rd3-e3 !	Kg7-g6 !	8	Re8-e4 ! Nb5-a7 !
4	Re3-e4	Kg6-h6	9	Re4-d4 ! Kg6-f6
5	Re4-f4 !	Na8-c7 !	10	Rd4-d8 !

and a sacrifice next move unless Black sacrifices first. The knight is driven into the open at move 5 (5...Kh7 loses more quickly), and all is then plain sailing.

This ending featured in a 1997 study by Fabrice Liardet which explored some of the winning positions for one White man (rook, bishop, or king) against king and knight.

Rook against two kings is straightforward. Apart from trivial wins for each side, there are some simple wins for the rook if it can crowd both defenders against the edge; otherwise the position is drawn. A longest win occurs with Rd2 against Kb8/Ka5; play 1 Rc2 Ka8 2 Rc3 and either 2...Ka7 3 Rc6 or 2...Ka6 3 Rc7.

There are three non-trivial positions in which the kings lose with the move but can draw without it (Ka8/Kd8 against Rf6, Ka8/Kb7 against Rd5, Kb8/Ka7 against Rd5).

3. Lone bishop

Nearly all endings with **bishop against two line-moving pieces** are trivial, and even the few which are not (for example, Bd1 to move against Qd6/Qd8) can be resolved very quickly. That said, even a short solution may have its charm. A delightful try line in a 1997 study by Fabrice Liardet comes down to wRb6/Rf8 to move against bBe1, and although White can sacrifice either rook he then leaves himself with a lost R v B ending!

Another interesting case is that of bishop against queen and bishop, where there are ten non-trivial positions in which the pieces lose with the move but can draw without it (Qa8/Ba6 against Bd2, Qb8/Bb6 against Bc2 or Be2, Qb8/Bb4 against Bc2, Qc8/Bc6 against Bf2, Qd8/Bd6 against Bc2, Qd8/Bd4 against Bc2, Qa7/Ba5 against Bd1, Qb7/Bb5 against Be1, Qc7/Bc5 against Bf1). In each of these positions, the queen is on a square of the same colour as her bishop, the bishop blocks the queen on the file, and the opposing bishop is on a square of the opposite colour six ranks away.

Bishop against queen and knight offers a single drawing position: Bd1 against Qd6/Nh8. White's only safe move is 1 Ba4, Black's only safe reply is 1...Qf4, and we have a reflection of the same position. There are also 228 non-trivial wins for the bishop to move and 25 non-trivial losses, but none is difficult. The longest win has Bb1 against Qb8/Nd8: play 1 Bh7 Qa7 2 Bg6 Nb7 3 Bf7, and Black has no good move.

There is one non-trivial position in which the side to move loses (Qb8/Nb6 against Bb1).

Bishop against rook and knight offers 392 non-trivial wins for the bishop to move and 43 non-trivial losses, but none is difficult. There are no draws.

There are two non-trivial positions in which the side to move loses (Ra8/Na5 against Bb1, Rb8/Nb6 against Bb1).

Bishop against bishop and knight, with unlike bishops, is always trivial; the player with the pieces, if not forced to make an immediate capture, can hold the draw by playing his knight to a square of the opposite colour to that of his bishop and leaving it there. With like bishops, all non-trivial positions are won for the lone bishop.

Bishop against two knights offers 3155 non-trivial wins for the bishop to move and 10 non-trivial losses. A longest win has Bd1 against Nb8/Ne7, with typical play 1 Bh5 Nc8 2 Bf3 Nd7 3 Be4 and the knights are dominated; hardly difficult, but White must avoid 2 Bg4 because 2...Nc6 3 BxN Na7 wins for Black. The non-trivial losses all feature a dominated bishop (for example, Bc1 against Nb7/Nd7).

There are nine non-trivial positions in which the side to move loses. The computer lists them as follows:

Na8/Ne6 v Be1	Nc8/Nf7 v Be1	Nb7/Nd7 v Bc1
	Nc8/Ne6 v Be1	
Nb8/Nd6 v Bd1		Nc7/Ne7 v Bd1
Nb8/Nc5 v Bh5	Nd8/Nd6 v Bd1	
	Nd8/Nf6 v Bf1	

The tenth non-trivial loss with White to move (Bc1 against Nd7/Na2) does not appear in this list because it is trivial with Black to play.

Bishop against king and queen offers 21 non-trivial wins for the bishop to move, 28 non-trivial losses, and 160 non-trivial draws. None is difficult, but in two cases (Bc1 or Be1 against Kd5/Qd7) a losing bishop can delay making a capture until move 4. The draws arise when the bishop can force the queen to sacrifice itself.

There are four positions in which the bishop loses with the move but can draw without it (Kd7/Qa7 against Bh7, Kd6/Qd7 against Bd2, Kd5/Qd7 against Bc1 or Be1) and three in which the same is true of the pieces (Kc8/Qb8 against Bh5, Kc7/Qa7 against Bg6, Kd6/Qd8 against Bc2).

Bishop against king and rook is generally similar, offering 20 non-trivial wins for the bishop to move, 61 non-trivial losses, and 300 non-trivial draws. In two cases, Bc1 or Be1 against Kd5/Rd7, a losing bishop can delay making a capture until move 8. In each case, 1 Bd2 is White's best move, and the computer then plays

1	......	Rd7-d8	4	Bd2-e1 !	Ke6-d5 !
2	Bd2-c1 !	Kd5-d6 !	5	Be1-d2 !	Rd8-d7 !
3	Bc1-d2 !	Kd6-e6	6	Bd2-e1	Rd7-e7 !

after which any White move allows Black to sacrifice both his men (but not by 7...Re1, which only draws). The play from Black's 1st move to his 5th is another "lose-a-move" manoeuvre.

There is one position (Kd7/Ra7 against Bh7) in which the bishop loses with the move but can draw without it.

Bishop against king and bishop is normally drawn when not trivially won or lost, but there are 35 non-trivial wins for the lone bishop. A longest has Bh7 against Ba8/Kc6, with typical play 1 Bg8 Kb7 2 Bf7 etc.

Bishop against king and knight is complicated in detail (the computer counts 4565 non-trivial wins for the bishop to move, 49 non-trivial losses, and 1107 non-trivial draws) but strategically straightforward.

- If the knight is on a square of the same colour as the bishop, the bishop can normally play to attack squares adjacent to the king, since a sacrifice of the king will leave Black with a losing N v B ending. The king will therefore have to retreat, and the bishop can hope to repeat the process and gradually drive him to the edge of

the board. For example, set Be1 against Nd6/Ke4, when White wins by 1 Bc3 Kf3 2 Bd2 Kg2 3 Be1 Kh1 4 Bf2, or against Ne7/Ke6, when the win is 1 Bc3 (Bg3 also wins) Kd7 2 Be5 Ke8 3 Bd6/Bf6. However, if the knight on one of the central 16 squares and the king is next to it, the king can run round the knight indefinitely and Black draws (for example, Be1 against Nd6/Kd5, play 1 Bc3 Kc6 2 Bd4 Kd7 3 Bb6 Ke6 and so on). White must also be careful lest the king sacrifice himself on c8 or d8 and come down to one of the exceptional *winning* N v B endings. Set Be1 against Nd6/Kf5 (our first example with bKf5 instead of e4), and 1 Bc3 fails; Black replies 1...Kg4 2 Bd2 Kh3, and now 3 Be1 actually loses (3...Kh4 4 Bxh4 Nc4). The only win is 1 Bh4 Ke4 2 Bf6 Kf3 3 Bg5 Kg2 4 Bh4 Kh1 5 Bf2, driving bK to h1 as before.

- If the knight is on a square of the opposite colour, advancing on the king is potentially dangerous, because Black may be able to sacrifice king and knight on successive moves. The bishop may even be dominated, any move leaving him liable to capture the knight. White's winning chances in this case are therefore few, and he may be hard pressed even to draw.

The computer gives Bd2 against Ke7/Na8 as a longest win for the bishop. Here the knight is on a square of the opposite colour to the bishop, but it is poorly placed in the corner and White can win in spite of this disadvantage. An optimal line of play is

1	Bd2-c3 !	Ke7-d7 !	4	Bb2-f6 !	Kb5-a6
2	Bc3-a1	Kd7-c6	5	Bf6-e7 !	Ka6-a7
3	Ba1-b2	Kc6-b5	6	Be7-c5 !	

with a striking final position (6...Nc7 7 BxK will be a win for the bishop). Several other waiting moves are as good at moves 2 and 3.

A longest win for the king and knight has Kd8/Nd7 against Bc1, with play

| 1 | Bc1-d2 ! | Kd8-c7 ! | 3 | Be1-d2 ! | Kd6-d5 ! |
| 2 | Bd2-e1 | Kc7-d6 ! | 4 | Bd2-e1 | Kd5-e4 ! |

and two sacrifices to follow. Here, even a square as distant from the centre as d7 allows the knight to exert a decisively cramping influence on the bishop, keeping it at bay until the king can come and help. White can play Bc1 instead of Be1 at moves 2 and 4, but the outcome is the same.

There are ten non-trivial positions in which the bishop loses with the move but can draw without it (Kd8/Nd7 against Bc1 or Be1, Kc7/Nd7 against Bd2, Kd6/Nd7 against Be1, Kd6/Nf7 against Be1, Kd5/Ne8 against Bd2, Kd5/Nf8 against Bd1, Kd5/Nd7 against Bd2, Kd5/Nh5 against Ba5, Kd5/Ng4 against Ba5) and five in which the same is true of the pieces (Ka8/Nf5 against Ba5, Ka8/Ng4 against Ba5, Ka8/Nh3 against Ba5, Kc8/Ne5 against Bc5, Kd8/Nf5 against Bd5).

This ending was examined by Fabrice Liardet in 1991 and 1997.

Bishop against two kings is normally drawn when not trivially won or lost, but there are 179 non-trivial wins for the bishop (mostly by threatening an immediate sacrifice which Black cannot evade) and 67 for the kings. No win for the bishop takes more than three moves, the longest having Ka8/Kb8 against a bishop able to play to a4 or e8; typical play is 1 Ba4 and either 1...Kaa7 2 Bc6 or 1...Kba7 2 Bd7.

In the absence of a quick win for the bishop, it is the kings which have such winning chances as exist, but the bishop can hold the draw if it can safely move to the long or an adjacent diagonal; it simply patrols this diagonal, and neither king can approach without allowing the bishop to sacrifice itself. The non-trivial wins for the kings all start with the bishop on c1, d2, e1, or an equivalent square. All such positions are won for the kings unless the bishop can sacrifice itself or move immediately to the long or an adjacent diagonal, the kings always being able to advance to a position of domination. A longest win has Kd8/Kd6 against Bd2, with typical play 1 Bh6 K8d7 2 Bc1 K7e6 3 Bd2 Ked5 4 Bc1 Kc4 5 Bh6 Kb3 (quicker than the apparently natural 5...Ke6) and the bishop must allow two sacrifices.

There is one non-trivial position (Kd8/Ka5 against Bd5) in which the kings lose with the move but can draw without it.

4. Lone knight

The knight is the weakest of the pieces in one-against-one endings (knight against knight depends on the move, knight against any other piece is normally a loss) and endings with a lone knight against two pieces are usually good for the pieces; unless the lone knight has a trivial win, or can threaten a sacrifice which the opponent cannot evade, the player with the two pieces cannot normally be prevented from sacrificing one of them and coming down to a winning ending with the other. In particular, all positions with **knight against two line-moving pieces** come into this category. There are however 13 non-trivial positions with knight against rook and bishop in which whoever is to move loses (Ba8/Rb7 against Nf4, Nh4, Ng3, or Nh2, Bb8/Rc7 against Ng4, Nf3,