An observation on an 8×8 knight's tour by Jaenisch

John Beasley, 17 March 2023

I recently came across a statement that all pandiagonal 4×4 magic squares can be obtained by setting *a*,*b*,*c*,*d* in the patterns below to 8,4,2,1 in some order, adding them up for each cell, and adding 1. Thus if we set *a*=8, *b*=4, *c*=2, *d*=1, we get the pandiagonal magic square on the right:

–	*a*	–	*a*		–	*b*	*b*	–		–	–	*c*	*c*		–	*d*	–	*d*		1	14	7	12
a	–	*a*	–		*b*	–	–	*b*		*c*	*c*	–	–		–	*d*	–	*d*		15	4	9	6
a	–	*a*	–		–	*b*	*b*	–		–	–	*c*	*c*		*d*	–	*d*	–		10	5	16	3
–	*a*	–	*a*		*b*	–	–	*b*		*c*	*c*	–	–		*d*	–	*d*	–		8	11	2	13

It occurred to me to wonder what might happen if we tried doing something like this with an 8×8 knight's tour. No such tour can be pandiagonally magic, the nearest being a tour published by C. F. Jaenisch in 1859 (12n in George Jelliss's catalogue). This tour has 180-degree rotational symmetry, each row and each column add to 260, the principal odd diagonal and each parallel broken odd diagonal to 256, and the principal even diagonal and each parallel broken even diagonal to 264. No other 8×8 knight's tour has all these properties.

27	30	51	40	53	2	15	42
50	39	28	31	14	41	54	3
29	26	37	52	**1**	56	43	16
38	49	**32**	25	44	13	4	55
23	36	45	12	57	**64**	17	6
48	11	24	**33**	20	5	58	61
35	22	9	46	63	60	7	18
10	47	34	21	8	19	62	59

The first step is to subtract 1 from each number and to express the result in binary:

011010	011101	110010	100111	110100	000001	001110	101001
110001	100110	011011	011110	001101	101000	110101	000010
011100	011001	100100	110011	000000	110111	101010	001111
100101	110000	011111	011000	101011	001100	000011	110110
010110	100011	101100	001011	111000	111111	010000	000101
101111	001010	010111	100000	010011	000100	111001	111100
100010	010101	001000	101101	111110	111011	000110	010001
001001	101110	100001	010100	000111	010010	111101	111010

We can now look at the occurrences of each power of 2:

–	–	32	32	32	–	–	32		16	16	16	–	16	–	–	–		8	8	–	–	–	–	8	8
32	32	–	–	–	32	32	–		16	–	16	16	–	–	16	–		–	–	8	8	8	8	–	–
–	–	32	32	–	32	32	–		16	16	–	16	–	16	–	–		8	8	–	–	–	–	8	8
32	32	–	–	32	–	–	32		–	16	16	16	–	–	–	16		–	–	8	8	8	8	–	–
–	32	32	–	32	32	–	–		16	–	–	–	16	16	16	–		–	–	8	8	8	8	–	–
32	–	–	32	–	–	32	32		–	–	16	–	16	–	16	16		8	8	–	–	–	–	8	8
32	–	–	32	32	32	–	–		–	16	–	–	16	16	–	16		–	–	8	8	8	8	–	–
–	32	32	–	–	–	32	32		–	–	–	16	–	16	16	16		8	8	–	–	–	–	8	8

–	4	–	4	4	–	4	–		2	–	2	2	–	–	2	–		–	1	–	1	–	1	–	1
–	4	–	4	4	–	4	–		–	2	2	2	–	–	–	2		1	–	1	–	1	–	1	–
4	–	4	–	–	4	–	4		–	–	–	2	–	2	2	2		–	1	–	1	–	1	–	1
4	–	4	–	–	4	–	4		–	–	2	–	2	–	2	2		1	–	1	–	1	–	1	–
4	–	4	–	–	4	–	4		2	2	–	2	–	2	–	–		–	1	–	1	–	1	–	1
4	–	4	–	–	4	–	4		2	2	2	–	2	–	–	–		1	–	1	–	1	–	1	–
–	4	–	4	4	–	4	–		2	–	–	–	2	2	2	–		–	1	–	1	–	1	–	1
–	4	–	4	4	–	4	–		–	2	–	–	2	2	–	2		1	–	1	–	1	–	1	–

The patterns for 8, 4, and 1 are very systematic, those for 32 and 16 are fairly systematic, and even that for 2 shows a fair amount of system when examined closely.

I have no idea how these patterns relate to the standard ways of constructing magic knight's tours.