# A class of four-move leaper tours self-complementary by diagonal reflection

John Beasley, 10 March 2023

A **four-move leaper** is a piece which, from any square on a chessboard, has precisely four moves available, a pair of complete tours of the board by a four-move leaper is called **complementary** if at each square one of the tours uses the two moves that the other tour does not, and a tour is called **self-complementary** if the complementary tour can be produced by rotation or reflection. This paper looks at some tours where the complementary tour is produced by diagonal reflection.

We consider a combined $(0,n)/(0,n+1)$ leaper on a square board of side $2n+1$. From the central square, this can leap to any edge; if one edge is nearer than the other, it has two moves towards the far edge but none towards the near edge. It therefore has precisely four moves available from any square, and so is a four-move leaper.

### Linear tours

Let us first consider the moves of such a leaper along a line. Suppose we have a line of squares numbered from 1 to $2n+1$, with a combined $n/n+1$ leaper at square 1. Let us suppose a sequence of leaps of length $n$ forwards or $n+1$ backwards (one of these will always be possible); then if we start at square 1 we shall visit squares $n+1$, $2n+1$, $n$, $2n$, $n–1$, $2n–1$, and so on in order, and after $2n$ leaps we shall be at square $n+2$ having visited every other square on the way. Now a final backwards leap of length $n+1$ will return us to our starting square.

There is a companion sequence whose leaps are of length $n+1$ forwards or $n$ backwards. From square 1, it visits the squares in reverse order, $n+2$, 2, $n+3$, 3, and so on, and after $2n$ leaps we are at square $n+1$ and a final backwards leap of length $n$ will take us back home.

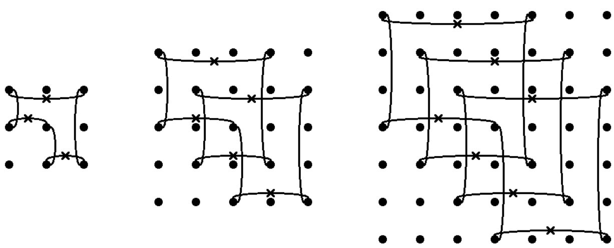
We shall make extensive use of sequences of $2n$ leaps of this kind. To diagram every leap is soon seen to be impracticable even if we show leaps of length $n$ on one side of the line and leaps of length $n+1$ on the other side; it is not too clear even for a line of length 5, and is quite unreadable for lines of length 7 or beyond:

Instead, we shall diagram only the further leap which would bring us back home, which we *don't* make, and we shall put a little cross on it to indicate that is it not made. So we shall diagram the sequences above as follows:

### Square-board tours

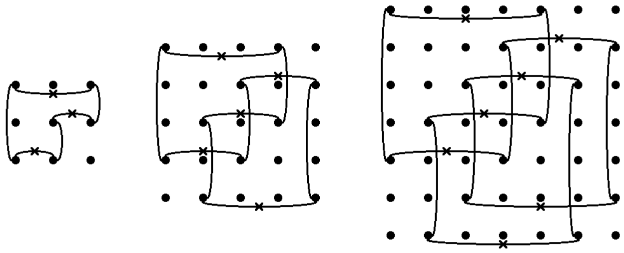
We now consider a square board of side $2n+1$, the rows being numbered $1\ldots2n+1$ from the top and the columns $1\ldots2n+1$ from the left. Our method will be to start at square (1,1), to perform a sequence of $2n$ horizontal leaps along row 1, to make one vertical leap, to perform a sequence of $2n$ horizontal leaps along the row we have now reached, to make another vertical leap, and to continue until we have returned to our starting point.

Let us first suppose that all our horizontal jumps are rightwards of length $n$ or leftwards of length $n+1$, and our vertical jumps downwards of length $n+1$ or upwards of length $n$. This causes us to visit the rows in order 1, $n+2$, 2, $n+3$, and so on, and each vertical jump brings us back to some square on the diagonal from (1,1) to $(2n+1,2n+1)$. This gives us a pattern like one of those shown below:



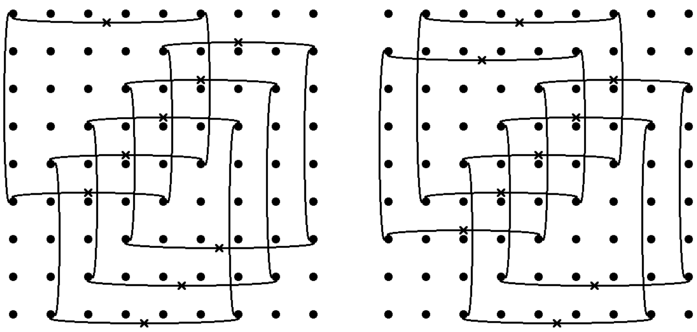
If we now look at row 1 and at the rightmost column $2n+1$, we see the leap from square $n+2$ to square 1 in row 1 is the one leap that we *don't* make, and that the leap from square $2n+1$ to square $n$ in column $2n+1$ is the one leap that we *do* make. The same can be shown to be true for every row $i$ and column $2n+2–i$, and a consequence is that if we reflect the tour in the diagonal from square $(1,2n+1)$ to square $(2n+1,1)$ we obtain the complementary tour.

Alternatively, we can make our vertical jumps of length $n$ downwards or $n$+1 upwards. This gives patterns like those shown below:



If we now look at row 1 and column 1, we see that the leap from square $n$+2 to square 1 in row 1 is again the leap that we don't make, but now it is the leap from square $n$+2 to square 1 in column 1 that we do make. A similar property can be shown to be true for every row $i$ and column $i$, so this time if we reflect the tour in the diagonal from square (1,1) to square (2$n$+1,2$n$+1) we obtain the complementary tour.

And there is more. The tour on the 5×5 board is also self-complementary by reflection in the diagonal from square (1,2$n$+1) to square (2$n$+1,1), though the other tours shown above are not. However, if we look at the tour on a 9×9 board (left-hand diagram below) and instead of starting from square (1,1) we start from square (1,2) as shown in the right-hand diagram, we see that the tour is again self-complementary by reflection in the diagonal from square (1,2$n$+1) to square (2$n$+1,1). It is also symmetric by 180-degree rotation.



All this generalizes, and we can say the following.

- If, given a combined (0,$n$)/(0,$n$+1) leaper on a square board of side 2$n$+1, we start at square (1,1), make 2$n$ horizontal leaps either of length $n$ rightwards or of length $n$+1 leftwards, then a vertical leap either of length $n$+1 downwards or of length $n$ upwards, and repeat until we have made a total of $(2n+1)^2$ moves, we have a complete tour of the board which is self-complementary by reflection in the diagonal from square (1,2$n$+1) to square (2$n$+1,1).

- If instead we make the vertical moves of length $n$ downwards or of length $n$+1 upwards, we have a tour which is self-complementary by reflection in the diagonal from square (1,1) to square (2$n$+1,2$n$+1).

- If $n$=2$k$ and we start from square (1,$k$) and do the same, we have a tour which is self-complimentary by reflection in either diagonal and has 180-degree rotational symmetry.