A class of self-complementary four-move leaper tours with horizontal symmetry

John Beasley, 1 March 2023

A **four-move leaper** is a piece which, from any square on a chessboard, has precisely four moves available (an example being the "fiveleaper", or combined (0,5)/(3,4) leaper, on an 8×8 board). A pair of complete tours of the board by a four-move leaper is called **complementary** if at each square one of the tours uses the two moves that the other tour does not. A tour is called **self-complementary** if the complementary tour can be produced by rotation or reflection.

This paper looks at tours which are self-complementary by 90-degree rotation (a 90-degree rotation produces the complementary tour, a further 90-degree rotation produces the original tour again) and which also have horizontal symmetry.

Background

We assume a square board of even side $2n$, and a tour which, like a knight's tour, covers the complete board and visits light and dark squares alternately. On such a board, the axis of horizontal symmetry is the vertical line between the two central files, and any two squares on the same horizontal line which are equidistant from it will be of different colour. A basic theorem now states that if such a tour on such a board is horizontally symmetric, it must include precisely two moves which are horizontal and are bisected by the axis of symmetry, and these must be at opposite ends of the tour.

For suppose that there is such a move, and that it joins squares A and A'. These squares will be on the same horizontal rank, A and A' will be of different colour, and the line joining them will be bisected by the axis of symmetry. Now let us move from A and A' to B and B', the moves A-A' and B-B' being horizontal reflections of each other. B and B' will again be on the same horizontal rank and equidistant from the axis of symmetry. Let us continue this until we reach squares Z and Z' from which no move can be made to a square we have not already visited. The only way we can continue the tour is to close it by joining Z to Z' (if they are a distance apart which allows a move between them to be made), and this gives us our second horizontal move bisected by the axis of symmetry.

Alternatively, let us suppose that there is no such move. We set up squares A and A' as before, on the same horizontal rank and equidistant from the axis of symmetry, and again continue until we reach squares Z and Z' from which no move can be made to a square we have not already visited. If the board is of side $2n$, the number of squares on it will be $4n^2$, and if the sequences $A\ldots Z$ and $A'\ldots Z'$ between them exhaust all the squares on the board then the length of each sequence will be $2n^2$. This is an even number, so the colours of A and Z will be different. But A and A' are also of different colour, so Z and A' will be of the same colour, and it will not be possible to continue the tour by joining them. It might be possible to join Z to A and Z' to A', but this would give us two separate half-tours and not one single complete tour.

Self-complementary four-move leaper tours on a 4×4 board

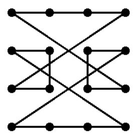
There are nine possible leapers on a 4×4 board: (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2,3), and (3,3). These can be combined to give five four-move leapers:

(0,1)/(0,3);
(0,1)/(2,3);
(0,3)/(1,2);
(1,2)/(2,3);
(1,1)/(1,3)/(3,3).

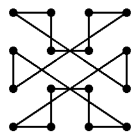
However, any move of a (1,1)/(1,3)/(3,3) leaper takes it to a square of the same colour, so such a leaper cannot tour the whole board.

The cases (0,1)/(0,3) and (1,2)/(2,3) can also be dealt with very quickly. In the case of the (1,2)/(2,3) leaper, no tour with horizontal symmetry is possible, because the leaper cannot make the two necessary horizontal moves. In the case of the (0,1)/(0,3) leaper, tours with horizontal symmetry exist, but no such tour can be part of a complementary pair. It will contain two and only two horizontal moves which are bisected by the axis of symmetry, but there are eight such moves available, so the complementary tour, which would also be horizontally symmetric, would have to contain the other six. This is impossible.

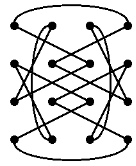
There remain the (0,1)/(2,3) and (0,3)/(1,2) leapers. The (0,1)/(2,3) leaper can make two self-complementary tours with horizontal symmetry, and the (0,3)/(1,2) leaper can make four:



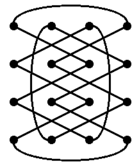
1	2	3	4
8	7	14	13
5	6	15	16
12	11	10	9



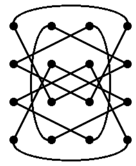
1	2	5	6
8	3	4	15
7	12	11	16
14	13	10	9



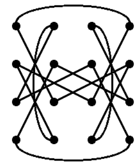
1	14	5	2
12	3	16	7
15	8	11	4
10	13	6	9



1	6	13	2
12	3	16	7
15	8	11	4
10	5	14	9



1	14	5	2
4	11	8	15
7	16	3	12
10	13	6	9



1	6	13	2
4	11	8	15
7	16	3	12
10	5	14	9

Magical properties

A tour is called **magic** if each row and each column adds to the same number (we'll look at diagonal properties in a moment). None of the above tours is magic, but the first (0,1)/(2,3) tour can be made magic by starting the numbering at 7, the second (0,3)/(1,2) tour by starting at 4, and the fourth (0,3)/(1,2) tour by starting at 4 or 8:

7	8	9	10
14	13	4	3
11	12	5	6
2	1	16	15

4	9	16	5
15	6	3	10
2	11	14	7
13	8	1	12

4	9	16	5
7	14	11	2
10	3	6	15
13	8	1	12

8	13	4	9
11	2	15	6
14	7	10	3
1	12	5	16

It will be seen that in every case each row and each column adds to 34. These magic tours were reported by George Jelliss in *Chessics* 26. Because of the rotational symmetry, they remain magic if every number is increased or decreased by 8.

Diagonal properties

A magic tour is called **pandiagonal** if each diagonal, plain or broken, adds to the number to which each row and column adds. A tour which visits light and dark squares alternately cannot be pandiagonal, because half the diagonals will consist of odd numbers only, the other half will consist of even numbers only, and the odd numbers from 1 to $4n^2$ add to a different sum from the even numbers. However, we can hope for a property which might be called "odd-even pandiagonal", in which each odd diagonal adds to one number and each even diagonal to a second number. The second and fourth of the tours above have this property:

36	32	36	32				
	↘	↘	↘	↘			
		4	9	16	5	←	34
	↗	15	6	3	10	←	34
36	↗	2	11	14	7	←	34
32	↗	13	8	1	12	←	34
36	↗	↑	↑	↑	↑		
32		34	34	34	34		

36	32	36	32				
	↘	↘	↘	↘			
		8	13	4	9	←	34
	↗	11	2	15	6	←	34
36	↗	14	7	10	3	←	34
32	↗	1	12	5	16	←	34
36	↗	↑	↑	↑	↑		
32		34	34	34	34		

They can therefore claim to be the most elegant tours of this kind.

Larger boards

No pair of complementary tours by a four-move leaper on a board larger than 4×4 can be horizontally symmetric. The argument is the same as that which was used for the (0,1)/(0,3) leaper on a 4×4 board: each tour of the pair must contain precisely two of the available horizontally symmetric moves, so there will be at least one left over.

The six tours shown at the top of the page are therefore the only tours of this kind.

These enumerations have been done by hand, and I am now 83. Please report any errors or omissions.