## On the taking of penalties

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The current Football World Cup, with its penalty shoot-outs to decide games still drawn at the end of extra time, has caused me to look at the mathematics of the taking of penalties.

A penalty is a competition between a penalty taker and a goalkeeper. The penalty taker may try to defeat the goalkeeper by the speed and accuracy of his shot; alternatively, he may try to send the goalkeeper diving the wrong way, or he may even shoot straight down the middle, with a view to scoring whichever way the goalkeeper dives (it has been done, and with success). Let us put a few specimen numbers to these possibilities.

Typically, a goalkeeper has a stronger and a weaker side. Let us suppose that if the taker shoots to his weaker side and he dives that way, he will stop one shot in three; that if the taker shoots to his stronger side and he dives that way, he will stop one shot in two; and that if the taker shoots straight down the middle and he stands his ground, he will stop everything. However, if he dives the wrong way, or stands his ground when he should have dived, or dives when he should have stood his ground, the taker will score. This gives the following tables:

| Goalkeeper's choice | Taker's choice: weak | strong | middle |
|---|---|---|---|
| Weak | 1/3 | 0 | 0 |
| Strong | 0 | 1/2 | 0 |
| Middle | 0 | 0 | 1 |

Goalkeeper's probability of success

| Taker's choice | Goalkeeper's choice: weak | strong | middle |
|---|---|---|---|
| Weak | 2/3 | 1 | 1 |
| Strong | 1 | 1/2 | 1 |
| Middle | 1 | 1 | 0 |

Taker's probability of success

The table on the left shows the goalkeeper's probability of success, that on the right the taker's.

On the face of it, the taker should always go for the goalkeeper's weaker side, when he will expect to score at least two out of three whatever the goalkeeper does. However, he can do better. According to the Theory of Games, he should choose between the goalkeeper's Weak side, Strong side, and Middle at random with probabilities $w$, $s$, $m$ such that his probability of success $p$ is the same whatever the goalkeeper does. If the goalkeeper dives to his weak side, we have

$$p = 2w/3 + s + m,$$

if he dives to his strong side, we have

$$p = w + s/2 + m,$$

if he stands his ground in the middle, we have

$$p = w + s,$$

and since $w$, $s$, and $m$ between them cover all possibilities, we have

$$w + m + s = 1.$$

If we solve these four equations, we find that $w = 1/2$, $s = 1/3$, $m = 1/6$, and $p = 5/6$. Only half the time should the taker go for the goalkeeper's weaker side, and by mixing his strategies he has increased his probability of success to 5/6.

We can perform a similar exercise for the goalkeeper, and if we do we find that $w = 1/2$, $s = 1/3$, $m = 1/6$, and $p = 1/6$. Thus the goalkeeper too should mix his strategies (unless he can work out from the taker's run-up which way the ball is going), and by doing so he can give himself a probability of success of 1/6 (and so restrict the taker's probability to the 5/6 we have just seen). We may notice that he should dive to his weaker side more often than to his stronger side.

We may also notice that the optimal strategy for the taker includes some shots down the middle. If he never shoots down the middle, the goalkeeper can dive to his weaker side with probability 3/5 and to his stronger side with probability 2/5, and reduce the taker's probability of success from 5/6 to 4/5.

Of course, these figures are only specimens, and we have not allowed for the possibility that the goalkeeper may guess wrong only for the shot to go past the post or over the bar. But the tables can be modified to allow for this, and while the details of the arithmetic will be different, there will be no difference of principle.