Some observations on a 1924 octagram board

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According to David Singmaster's *Sources in Recreational Mathematics*, a 1924 book *Fun, Mirth & Mystery* by R. A. Hummerston includes a peg solitaire game played on an octagram board. Examination shows this game to present some interesting features. I haven't seen the original source, and am relying on David's description.

The game, called "Perplexity", uses a 16-hole board

1 2
3
4 5 6 7
8 9
10 11 12 13
14
15 16

where jumps are allowed along the rows, columns, and diagonals. The given task is to leave hole **16** vacant and play to leave a single survivor, and a further challenge is to specify where the final survivor is to be left. Even this is not particularly difficult, but the game has much more to offer. According to the computer, the only single-vacancy single-survivor problem that cannot be solved is "vacate and finish at **3**". We can vacate **3** and finish at any other hole, and we can vacate any hole other than **3** (or **8**/**9**/**14**) and finish wherever we choose.

To see why "vacate and finish at **3**" cannot be solved, let us start by marking off the diagonals in the usual way:

A B
B
A B C A
B A
B C A B
A
A B

We count the numbers of men in each diagonal, and we say that two diagonals are "in phase" if the parities of the numbers of men in them are the same, and "out of phase" if they are different. With three diagonals, either all three are in phase with each other or one is out of phase with the other two.

Now let us look at the differences between the initial and target positions. We count the numbers of men in each diagonal, and if the difference for *A* is out of phase with those for *B* and *C* we must bring it into phase in the course of the solution by making an odd number of jumps along the diagonal **7**-**15**. Similarly, if the difference for *B* is out of phase we must bring it into phase by making an odd number of jumps along the diagonal **2**-**10**, and if the difference for *C* is out of phase we must make odd numbers of jumps along both diagonals. If the differences for *A*, *B*, *C* are all in phase then we cannot make an odd number of jumps along either diagonal, though we can make even numbers of jumps along them. In the problem "vacate and finish at **3**", the differences between the initial position (all except **3** occupied) and the final position (only **3** occupied) are 7 x *A*, 5 x *B*, 2 x *C*, and since that for *C* is out of phase we must make an odd number of jumps along each of the diagonals **2**-**10** and **7**-**15**. Similarly, we must make an odd number of jumps along each of the diagonals **1**-**13** and **4**-**16**.

These jumps may be either by pegs **3**/**8**/**9**/**14** (we shall call these "key pegs") or over them. However, each jump over a key peg reduces the number of key pegs by 1, we start with 3 and must finish with 1, and so we must make precisely 2 jumps of this kind. The remaining jumps along the diagonals must be by the key pegs.

(a)	(b)	(c)	(d)	(e)	(f)
–1	1	1	1	2	1
O O	*E E*	*E E*	*O O*	*E O*	*O O*
1 1	– –	1 –	1 –	– –	–1 1
O O	*O O*	*E O*	*O E*	*O E*	*O O*
1	1	–	–	–	1

Ignoring the eight outside holes, the task to be fulfilled can be represented by (a) above. Here "–1" indicates that there is a deficiency in this hole that is to be filled, "1" that there is a surplus to be liquidated, and *"O"* that there are to be an odd number of jumps along this diagonal (we shall use *"E"* similarly to indicate an even number of jumps). To fix our ideas, let the first jump of a solution be **8**-**3**, and let us first suppose that the final jump is **9**-**3**. This leaves (b) as the task remaining. Let us first consider the jumps by the key pegs. After one such jump, we are left with a task of type (c) or (d) (we ignore rotations and reflections). After a second, we have either another task of type (b) or one of type (e). In each case, a third jump will bring us to another task of type (c) or (d), and so it will go on. We are never left with a task of any other kind.

Eventually, we run out of jumps by key pegs, and since there will have been an even number of them we shall have a task of type (b) or (e). This must now be fulfilled by two jumps over the key pegs where they stand, which is quickly seen to be impossible (for example, if in (b) we play **16**-**11** to liquidate the peg at **14**, we cannot then realise the *O* at **12**). Hence there can be no solution of this kind.

Alternatively, suppose that the final jump is **8**-**3**. We now have task (f) to be fulfilled, and the necessary jump **3**-**8** or **14**-**8** to deal with the deficiency at **8** leaves a task of type (d). This takes us back into the previous analysis, so there can be no solution of this kind either.

We leave "vacate **3**, finish at **8**" and "vacate **3**, finish at **14**", which can be solved, to the reader.