Notes on a puzzle by Conway and O'Beirne

John Beasley, 25/26 November 2016, footnote added 1 December

Some thirty years ago, I was given the following puzzle: Fill a 5x5x5 cube with five unit cubes, six 1x2x4 planks, and six 2x2x3 columns. Thinking about it more recently, and wondering about its authorship, I looked for it in David Singmaster's *Sources in Recreational Mathematics*, and to my surprise I failed to find any reference to it. I therefore posted a request for help on this site.

Several readers came forward with information. In particular, Dic Sonneveld, who had made a major contribution to my update to the history of peg solitaire, sent me a number of references and pictures of practical realisations, the most important being to a recent book *The Mathematics of Various Entertaining Subjects* edited by Jennifer Beineke and Jason Rosenhouse. This book contains papers presented at a conference on recreational mathematics held in 2013, among which was one by Derek Smith discussing a generalization of this puzzle. Early realisations of the puzzle in its basic 5x5x5 form tended to give credit for its authorship to John Conway, but Smith, on the authority of a private communication from Conway in 2007, states that Conway shares credit for it with O'Beirne. So "a puzzle by Conway and O'Beirne" would seem to be the correct attribution.

(Thomas O'Beirne was a leading light in the British puzzle fraternity, making several contributions to the magazine *New Scientist* in the 1960s and having a book *Puzzles and Paradoxes* published by OUP in 1965. James Dalgety describes him as a very underrated puzzle inventor. On the evidence of the references to it in David Singmaster's *Sources*, his *Puzzles and Paradoxes* would seem to deserve a reprint.[*])

Although originally set in 5x5x5 form, the puzzle generalizes. Given a cube of odd side n, we can construct it from ($4n$–3) pieces as follows:

n unit cubes
six 2x1x(n–1) planks
six 2x2x(n–2) columns
six 2x3x(n–3) platters
...
six 2x($\{n$–1$\}$/2)x($\{n$+1$\}$/2) platters

with appropriate omissions for small values of n (for example, a 5x5x5 cube requires planks and columns only). Smith, working from the outside in, gives a proof that a solution exists for all odd n and that it is unique to within rotation and reflection. However, while waiting for my local bookshop to obtain the book, I did my own analysis working from the inside out, and found a simple and systematic way of getting from a solution for a cube of side n to one for a cube of side n+2. Although it does not address the question of uniqueness, I think this presentation helps to clarify the problem. Smith had clearly done a similar analysis himself (as no doubt had Andy Liu, whom he thanks for introducing him to the generalized forms of the puzzle), but space in books is limited, and he confines himself to his outside-in proof of uniqueness.

The present paper will therefore comprise

- a solution to the basic 5x5x5 problem,
- a demonstration by induction of a solution for a cube of any odd side n,
- a few observations on how easy or otherwise the problem is in isolation.

A proof that the solutions so found are unique to within rotation and reflection will be found in Smith's paper.

[*] David Singmaster tells me that it was reprinted by Dover in 1984.

Solution of the basic 5x5x5 puzzle

Let U...U be the unit cubes, A1...A6 the 2x2x3 columns, and B1...B6 the 2x1x4 planks. Then we have

U	B1	B1	B1	B1
B2	B1	B1	B1	B1
B2	A2	A2	A6	A6
B2	A2	A2	A6	A6
B2	A2	A2	B6	B6

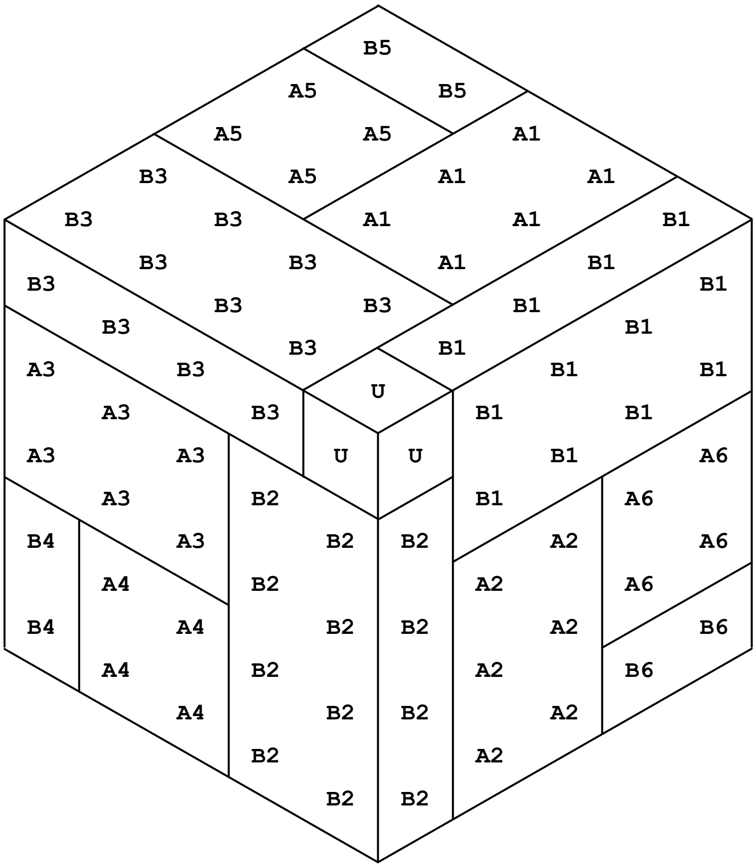
B3	B3	A1	A1	A1
B2	U	A1	A1	A1
B2	A2	A2	A6	A6
B2	A2	A2	A6	A6
B2	A2	A2	B6	B6

B3	B3	A1	A1	A1
A3	A3	A1	A1	A1
A3	A3	U	A6	A6
A4	A4	A4	A6	A6
A4	A4	A4	B6	B6

B3	B3	A5	A5	B5
A3	A3	A5	A5	B5
A3	A3	A5	A5	B5
A4	A4	A4	U	B5
A4	A4	A4	B6	B6

B3	B3	A5	A5	B5
A3	A3	A5	A5	B5
A3	A3	A5	A5	B5
B4	B4	B4	B4	B5
B4	B4	B4	B4	U

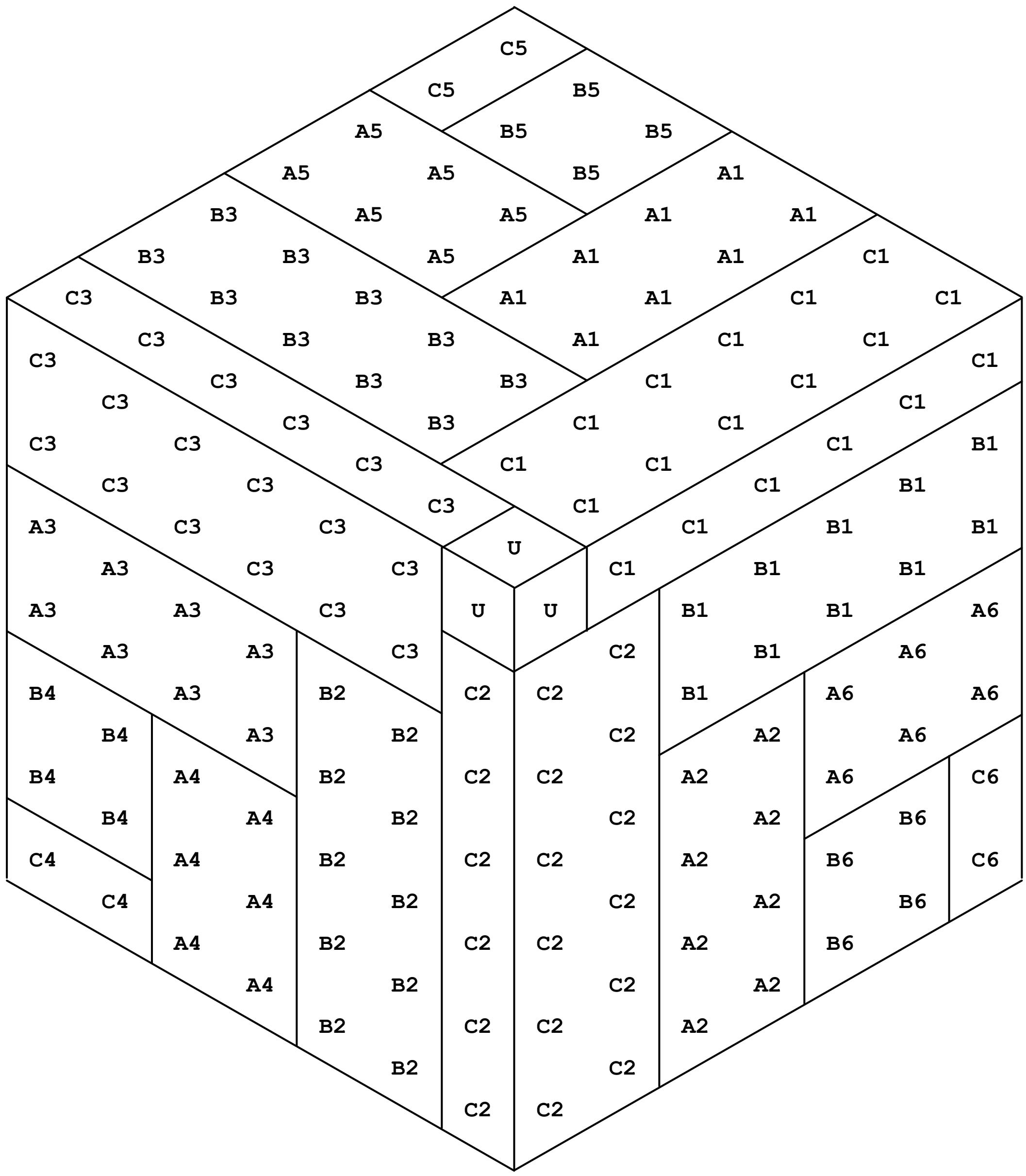
slice by slice from the front. The unit cubes are strung out along a diagonal of the 5x5x5 cube, and the remaining shapes are arranged around this diagonal in a pattern with three-fold rotational symmetry and diametric symmetry (B1/B2/B3 rotate clockwise to give B2/B3/B1, A1/A2/A3 similarly, A4 is diametrically opposite to A1, and so on). This becomes clearer if we look down at the corner surrounded by B1/B2/B3 :



The rotational symmetry is now obvious.

Moving from *n* to *n*+2

If we look at the diagram above, we note that six of the edges of the 5x5x5 cube (the three facing us and the three at the back which we can't see) are formed by unit cubes and planks. The remaining six edges each comprise one section of length 2 from a plank, one section of length 2 from a column, and one section of length 1 from another plank, and we see that each column and each plank has contributed a section of length 2 to one edge and to one edge only. We can therefore convert this into a 7x7x7 solution by expanding each 2x2x3 column into a 2x3x4 platter and each 2x1x4 plank into a 2x2x5 column, and adding two more unit cubes and six new 2x1x6 planks to complete the outermost shell. This produces the diagram on the next page.



Here, A1...A6 have become platters, B1...B6 have become columns, and C1...C6 are the new planks. We may also notice that the orientation of what Smith calls the "propellor" around the unit cells has flipped. If we look at a face of the 5x5x5 cube on the previous page, rotated so that the unit cell is at the top left corner, we see that the top plank is face-on and the plank to the left is edge-on. If we do the same with the 7x7x7 cube above, the top plank is edge-on and the plank to the left is face-on.

This clearly generalizes. At each stage, we replace the 2x(j)x(k) platters by 2x(j+1)x(k+1) platters, the 2x2x(n–2) columns by 2x3x(n–1) platters, and the 2x1x(n–1) planks by 2x2x(n) columns, and we add two new unit cubes and six new 2x1x(n+1) planks to complete the outermost shell. Proceeding thus from the 7x7x7 cube above to a 9x9x9 cube gives the solution shown in slice-by-slice form on the next page.

U	D1	D1	D1	D1	D1	D1	D1	D1
D2	D1	D1	D1	D1	D1	D1	D1	D1
D2	C2	C2	B1	B1	B1	B1	B1	B1
D2	C2	C2	B1	B1	B1	B1	B1	B1
D2	C2	C2	A2	A2	A6	A6	A6	A6
D2	C2	C2	A2	A2	A6	A6	A6	A6
D2	C2	C2	A2	A2	B6	B6	C6	C6
D2	C2	C2	A2	A2	B6	B6	C6	C6
D2	C2	C2	A2	A2	B6	B6	D6	D6

D3	D3	C1	C1	C1	C1	C1	C1	C1
D2	U	C1	C1	C1	C1	C1	C1	C1
D2	C2	C2	B1	B1	B1	B1	B1	B1
D2	C2	C2	B1	B1	B1	B1	B1	B1
D2	C2	C2	A2	A2	A6	A6	A6	A6
D2	C2	C2	A2	A2	A6	A6	A6	A6
D2	C2	C2	A2	A2	B6	B6	C6	C6
D2	C2	C2	A2	A2	B6	B6	C6	C6
D2	C2	C2	A2	A2	B6	B6	D6	D6

D3	D3	C1	C1	C1	C1	C1	C1	C1
C3	C3	C1	C1	C1	C1	C1	C1	C1
C3	C3	U	B1	B1	B1	B1	B1	B1
B2	B2	B2	B1	B1	B1	B1	B1	B1
B2	B2	B2	A2	A2	A6	A6	A6	A6
B2	B2	B2	A2	A2	A6	A6	A6	A6
B2	B2	B2	A2	A2	B6	B6	C6	C6
B2	B2	B2	A2	A2	B6	B6	C6	C6
B2	B2	B2	A2	A2	B6	B6	D6	D6

D3	D3	B3	B3	A1	A1	A1	A1	A1
C3	C3	B3	B3	A1	A1	A1	A1	A1
C3	C3	B3	B3	A1	A1	A1	A1	A1
B2	B2	B2	U	A1	A1	A1	A1	A1
B2	B2	B2	A2	A2	A6	A6	A6	A6
B2	B2	B2	A2	A2	A6	A6	A6	A6
B2	B2	B2	A2	A2	B6	B6	C6	C6
B2	B2	B2	A2	A2	B6	B6	C6	C6
B2	B2	B2	A2	A2	B6	B6	D6	D6

D3	D3	B3	B3	A1	A1	A1	A1	A1
C3	C3	B3	B3	A1	A1	A1	A1	A1
C3	C3	B3	B3	A1	A1	A1	A1	A1
A3	A3	A3	A3	A1	A1	A1	A1	A1
A3	A3	A3	A3	U	A6	A6	A6	A6
A4	A4	A4	A4	A4	A6	A6	A6	A6
A4	A4	A4	A4	A4	B6	B6	C6	C6
A4	A4	A4	A4	A4	B6	B6	C6	C6
A4	A4	A4	A4	A4	B6	B6	D6	D6

D3	D3	B3	B3	A5	A5	B5	B5	B5
C3	C3	B3	B3	A5	A5	B5	B5	B5
C3	C3	B3	B3	A5	A5	B5	B5	B5
A3	A3	A3	A3	A5	A5	B5	B5	B5
A3	A3	A3	A3	A5	A5	B5	B5	B5
A4	A4	A4	A4	A4	U	B5	B5	B5
A4	A4	A4	A4	A4	B6	B6	C6	C6
A4	A4	A4	A4	A4	B6	B6	C6	C6
A4	A4	A4	A4	A4	B6	B6	D6	D6

D3	D3	B3	B3	A5	A5	B5	B5	B5
C3	C3	B3	B3	A5	A5	B5	B5	B5
C3	C3	B3	B3	A5	A5	B5	B5	B5
A3	A3	A3	A3	A5	A5	B5	B5	B5
A3	A3	A3	A3	A5	A5	B5	B5	B5
B4	B4	B4	B4	B4	B4	B5	B5	B5
B4	B4	B4	B4	B4	B4	U	C6	C6
C4	C4	C4	C4	C4	C4	C4	C6	C6
C4	C4	C4	C4	C4	C4	C4	D6	D6

D3	D3	B3	B3	A5	A5	C5	C5	D5
C3	C3	B3	B3	A5	A5	C5	C5	D5
C3	C3	B3	B3	A5	A5	C5	C5	D5
A3	A3	A3	A3	A5	A5	C5	C5	D5
A3	A3	A3	A3	A5	A5	C5	C5	D5
B4	B4	B4	B4	B4	B4	C5	C5	D5
B4	B4	B4	B4	B4	B4	C5	C5	D5
C4	C4	C4	C4	C4	C4	C4	U	D5
C4	C4	C4	C4	C4	C4	C4	D6	D6

D3	D3	B3	B3	A5	A5	C5	C5	D5
C3	C3	B3	B3	A5	A5	C5	C5	D5
C3	C3	B3	B3	A5	A5	C5	C5	D5
A3	A3	A3	A3	A5	A5	C5	C5	D5
A3	A3	A3	A3	A5	A5	C5	C5	D5
B4	B4	B4	B4	B4	B4	C5	C5	D5
B4	B4	B4	B4	B4	B4	C5	C5	D5
D4	D4	D4	D4	D4	D4	D4	D4	D5
D4	D4	D4	D4	D4	D4	D4	D4	U

This shows the progression in another way. The middle three rows and columns of the middle three slices give a solution to "fill a 3x3x3 cube with three unit cubes and six 1x2x2 squares", a problem due to Conway which is in chapter 24 of *Winning Ways for your Mathematical Plays*, the middle five rows and columns of the middle five slices give the 5x5x5 solution with which we started, and the middle seven rows and columns of the middle seven slices give a slice-by-slice representation of the 7x7x7 solution we have just seen.

We may also note the evolution of B1...B6. They don't contribute at all to the 3x3x3 cube, in the 5x5x5 they appear as 2x1x4 planks, in the 7x7x7 as 2x2x5 columns, and in the 9x9x9 as 2x3x6 platters. They will become successively larger platters as the size of the cube increases further.

Smith has cornerwise pictures of a 9x9x9 cube, and anyone comparing the two will notice that his solution has the opposite propellor orientation; with the small cube in the top left corner, his top plank is edge-on and his left plank face-on.

How difficult is the original 5x5x5 problem when attempted in isolation?

It seems to depend on whom you ask. Dic Sonneveld gave me references to a dealer's blurb which described it as "one of the most difficult games out there" and gave it a difficulty rating of 4.5 on a scale from 1 to 5, and to a reviewer of another incarnation who admitted failure after more than a week. Yet I remember solving it mentally while waiting for a concert to start, and even then I was far from being the world's fastest puzzle solver. Why so great a disparity?

I think there are two reasons. Firstly, I attacked the problem with a measure of perspicacity (or knowledge, or experience, or whatever); secondly, I was lucky. The perspicacity (or whatever) led me quickly to the realisation that each of the five unit cubes had to be in a different slice whether top to bottom, or left to right, or front to back. So I tried the most obvious way of achieving this, which was to put them along one of the cube's internal diagonals, and everything fell into place very quickly. Had I not been fortunate enough to approach the problem in this way, I might well have found myself as bewildered as the reviewer.

Summary

We have shown that the problem can be solved for a cube of any odd size. We have not shown the solutions to be unique (give or take the two possible orientations of the propellors), and for this the reader is referred to a paper by Derek Smith in *The Mathematics of Various Entertaining Subjects*. This strikes me as a first-rate piece of work, elegant, ingenious, and lucidly presented, and I am very happy to have been able to draw attention to it.

Acknowledgements

Above all, to Dic Sonneveld, not just for the references cited above but for links to many other blurbs, reviews, and blogs. But I am also grateful to James Dalgety, whose observations on the relation to Conway's 3x3x3 puzzle started me on the road to a systematic solution, and to Jerry Slocum, who sent me a photograph of a 7x7x7 cube made by Richard Thiessen (item 30308 in his collection) which reassured me that I was on the right lines.