Complementary tours on four-branch networks

John Beasley, 23 May - 5 June 2023, afterthought added 15 June

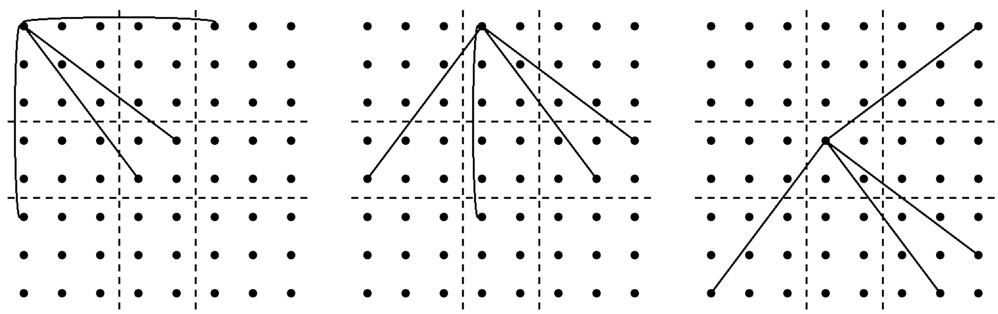
A **network** is a set of points joined by lines, a **four-branch network** is a network in which each point has precisely four lines meeting at it, and a **tour** is a closed circuit which visits each point once and once only. Tours will be regarded solely as geometrical patterns, no account being taken of starting point or direction of travel, and tours which can be converted into each other by rotation or reflection will be regarded as identical. A pair of tours on a four-branch network is **complementary** if at every point one tour uses the two branches that the other does not, and a tour is **self-complementary** if the complementary tour can be obtained by rotation or reflection.[1]

This paper will examine complementary and self-complementary tours on various networks: networks on a chessboard, which we shall regard as an 8 × 8 square array of points, networks on a 4 × 4 square array, 8-point networks, networks on the vertices of regular or semi-regular convex solids, a class of indefinitely large networks, and networks on which no pairs of complementary tours exist. I do not expect anything that follows to be new, but it has seemed worth while to bring it all together under one roof.

Networks on 8 × 8 square array

The classic tour on an 8 × 8 chessboard is the "Knight's Tour", in which a knight visits each of the 64 squares in turn and ends up back at its starting point. This has been a subject of study since at least the ninth century. It can be regarded as a tour of a network within an 8 × 8 square array of points, the touring man always moving like a knight, two points in one direction and one in the other, and the branches of the network being the knight moves available from each point. However, the number of options open to a knight depends on its position on the board, varying from eight if it is on one of the sixteen central squares down to two if it is in a corner, so the number of branches at a point in the corresponding network varies from eight down to two, and we are interested here in networks in which every point has the same number of branches.

A knight moves two squares in one direction and one in the other, giving a total move length, measured from square centre to square centre, of the square root of five. Now suppose that instead of making knight moves we always move to a point a distance exactly five points away: five in one direction and none in the other, or four in one direction and three in the other. On an 8 × 8 square array, we now do have four branches at every point. At a point in the corner, and indeed at any point not on one of the two central ranks or files, there are two orthogonal (horizontal or vertical) and two slanting branches; at a point lying on one of the two central ranks or files but not being one of the four central points, there are one orthogonal and three slanting branches; at one of the four central points, there are four slanting branches.



It is therefore natural to ask (a) can such a network be toured, and (b) given that there are four branches at each node, can there be two tours such that each uses the branches that the other does not?

[1] I am told that the correct mathematical term for "tours on four-branch networks" is "Hamiltonian circuits on quartic graphs", but my knowledge of mathematical jargon was imperfect even in my student days and has certainly not improved since. This is an informal paper, and I shall use ordinary everyday terminology as far as possible.

Networks on an 8 × 8 square array (continued).

The first question, of the possible existence of a tour of this network, appears first to have been answered by Maurice Kraitchik, who published one in 1927 in his book *Le Problème du Cavalier*. These tours do not lend themselves to graphical presentation, and I shall content myself with numbering the points traversed in order (his tour is the left-hand tour below, possibly reflected to start in the top left-hand corner):

1	10	55	20	37	2	9	54
42	25	50	7	30	43	24	49
39	4	17	58	63	12	5	18
28	61	22	53	46	27	60	21
31	14	35	48	41	32	15	36
64	11	56	19	38	3	8	57
45	26	51	6	29	44	23	52
40	33	16	59	62	13	34	47

1	56	31	22	13	46	3	54
42	33	20	17	52	11	44	35
27	62	49	40	37	28	9	60
14	47	4	7	**64**	57	30	25
23	12	45	2	55	32	21	16
18	51	10	43	34	19	50	53
41	38	29	26	61	48	39	36
6	63	58	15	24	5	8	59

What appears not to have been noticed until George Jelliss spotted it a few years ago is that this tour also answers the second question, the tour on the right using precisely the branches that the tour on the left does not. George says that there is no mention of this in the book; is it possible that Kraitchik overlooked it?

On reflection, yes, I think this quite possible. Nowadays, questions like this can be answered by telling a computer to go find, good dog, when it scampers happily away and in due course lays at our feet a large pile of bones, sticks, tennis balls, or whatever. In Kraitchik's day, the work had to be done by hand, and it was natural to look at the simplest and most straightforward possibilities first. There are 128 branches in the network, 48 of them being orthogonal, and if we look at Kraitchik's tour we see that no fewer than 36 of the orthogonal branches are used. In six of the nine four-branch squares on the board (1-2-3-64, 10-9-8-11, 55-54-57-56, 42-43-44-45, 25-24-23-26, 50-49-52-51) three of the four orthogonal branches are used (to have used all four would have given a loop, which no tour can contain); in the other three four-branch squares (39-12-13-40, 4-5-34-33, 17-18-47-16) two of the orthogonal branches are used; all the orthogonal branches along the central files and ranks are used. Furthermore, the unused orthogonal branches show a fair amount of pattern. 42-45 and 49-52 mirror each other vertically, as do 4-33 and 5-34; 54-57 and 40-13 mirror each other diagonally, as do 16-47 and 18-47; 39-12 and 64-3 mirror each other horizontally; 11-8 and 26-23 are parallel and adjacent.

What Kraitchik appears to have done (I hope I am not misrepresenting him) is to have put as many orthogonal branches as possible into a trial tour, and then to have adjusted the rest until everything fitted. He having thus found a solution by using as many orthogonal branches as possible, I find it quite feasible that he did not notice, or at least did not notice until after the book had gone to press, that the unused branches, which necessarily included as *few* orthogonal branches as possible, also gave a tour.

Be all this as its may, Kraitchik's work remained largely unknown in England, and although some other isolated tours were published, people continued to wonder whether a complementary pair of tours was possible. In 1991, George Jelliss raised the question in *Variant Chess*, which prompted an investigation by Tom Marlow. Marlow asked his computer to tell him if there was a tour where the complementary tour could be found by 180-degree rotation, and it duly reported the tour shown in the left-hand table below.

1	60	15	8	53	4	27	18
32	41	24	37	12	33	40	25
51	48	35	30	43	58	49	10
54	5	28	45	**64**	61	56	7
13	22	39	26	19	14	23	38
2	59	16	9	52	3	36	17
31	42	57	50	11	34	29	44
20	47	62	55	6	21	46	63

63	46	21	6	55	62	47	20
44	29	34	11	50	57	42	31
17	36	3	52	9	16	59	2
38	23	14	19	26	39	22	13
7	56	61	**64**	45	28	5	54
10	49	58	43	30	35	48	51
25	40	33	12	37	24	41	32
18	27	4	53	8	15	60	**1**

If this table is rotated through 180 degrees we get the right-hand table, and the two tours are indeed complementary.

Networks on an 8 × 8 square array (continued).

Thinking along similar lines, in 2010 I asked my own computer to see if there were any tours where a rotation by 90 degrees rather than 180 would give the complementary tour, and it reported that there were no fewer than 224. This figure should be regarded as provisional until it has been confirmed by an independent worker, since if a programming or machine error had caused me to miss a batch there would have been nothing to tell me, but I like to think it will stand up.

However, none of these tours has any particular attraction apart from the fact of its existence, and in particular none has horizontal or vertical symmetry. The reason is not far to seek. Let us consider horizontal symmetry, where the axis of symmetry is the vertical axis between the two central columns. There are eight branches which are perpendicularly bisected by this axis, so one tour or the other must contain at least one of them. Let us proceed from the two points at the ends of this branch simultaneously, always keeping in step. The tour being horizontally symmetric, this produces a sequence of pairs of points which are on the same horizontal line as each other and equidistant from the axis of symmetry, and we can terminate this process only by joining the two current ends. This produces a closed loop with the two branches perpendicularly bisected at opposite ends of it, and if this loop forms a complete tour the other six branches perpendicularly bisected must have been missed out.

But there are eight branches on the board which are so bisected, and if one tour contains at most two of them, the complementary tour must contain the other six. But the complementary tour will also be symmetric, and so it too can only contain two of them So four must be left over from both tours, and we have an impossibility.

As to what this means in practice, consider the tour in the left-hand table below, which was found by George Jelliss (I have rotated to bring the starting point to the top rank).

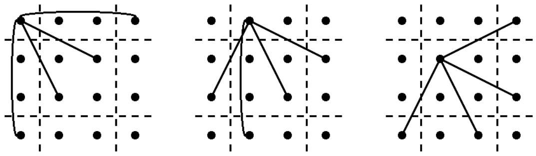
62	**1**	40	55	10	25	**64**	3
17	38	47	12	53	18	27	**48**
36	45	14	5	60	51	20	29
43	24	7	34	31	58	41	22
54	9	26	63	2	39	56	11
61	52	19	28	37	46	13	4
16	59	50	21	44	15	6	**49**
35	**32**	57	42	23	8	**33**	30

48	29	14	35	26	47	32	13
37	**1**	**62**	**44**	**17**	**59**	**4**	24
42	**5**	20	51	10	41	**56**	19
27	**57**	33	12	49	28	**64**	34
8	**60**	54	23	38	7	**61**	53
45	**30**	15	36	25	46	**31**	16
50	**2**	**63**	**43**	**18**	**58**	**3**	11
39	6	21	52	9	40	55	22

This has both horizontal and vertical symmetry, and I have put the end points of the branches bisected by the axes of symmetry in bold. If we now consider the complementary moves, which are shown in the right-hand table, we see that they form not a single tour but three separate loops: a simple square from 1 to 4, a more complicated loop from 5 to 56, and a final octagram from 57 to 64. Something like this always happens; if a tour on an 8 × 8 board has horizontal or vertical symmetry, the complementary moves do not form a tour, they form a set of disconnected loops.

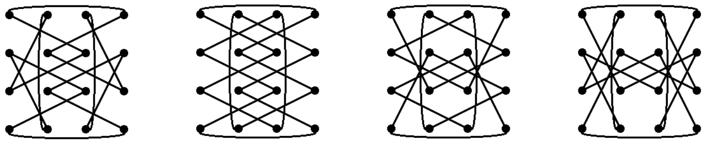
Networks on a 4 × 4 square array

If on a 4 × 4 square array we have "knight's move" branches and also orthogonal branches of length three, we again have four branches at every point. At a point in the corner, there are two orthogonal and two slanting branches; at an edge point away from the corners, there are one orthogonal and three slanting branches; at one of the four central points, there are four slanting branches.

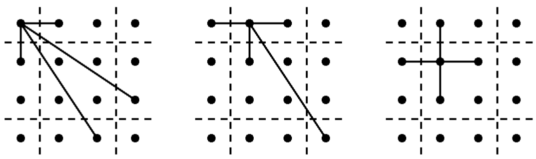


Now, however, there are only four branches which are perpendicularly bisected by the axis of horizontal symmetry, and we can hope to find pairs of complementary tours which are horizontally symmetric and use two of them each. The same is true of vertical symmetry.

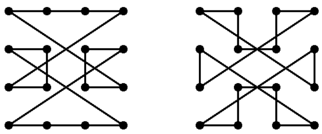
There are in fact four tours which have both horizontal and vertical symmetry and whose complementary tours can be obtained by reflection about either diagonal or by 90-degree rotation.



Another way of obtaining a four-branch network on a 4 × 4 square array is to have one-step orthogonal branches and "extended knight's move" branches (three steps in one direction and two in the other).

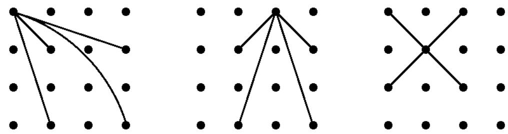


There are now two tours which have both horizontal and vertical symmetry and whose complementary tours can be obtained by reflection about either diagonal or by 90-degree rotation.



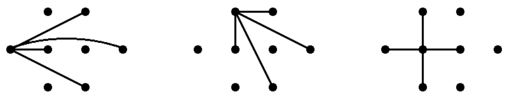
8-point networks

Yet another way of obtaining a four-branch network on a 4 × 4 square array is to have a one-step diagonal branch, a three-step diagonal branch, and a “diagonal knight’s move” branch (two steps in one diagonal direction and one in the other).

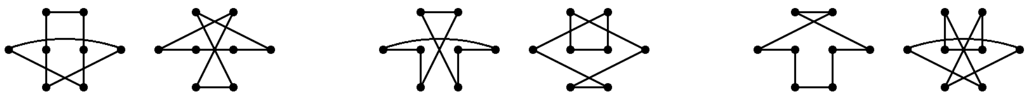


However, if we start at the top left-hand corner, we find that we can only reach half the points of the array; we can reach the first and third points in the first and third rows and the second and fourth points in the second and fourth rows, but no others. So we don’t have a 16-point network, but only an 8-point network.

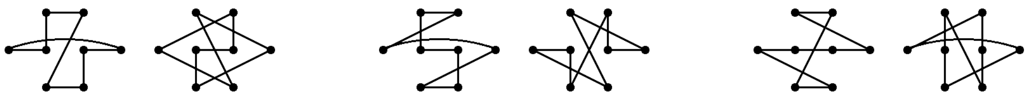
That said, this 8-point network is not without interest. It is convenient to rotate it through 45 degrees and to depict it as a lozenge-shaped array with one-step, three-step, and “knight’s move” branches.



There are six tours with horizontal symmetry, and they form three complementary pairs:



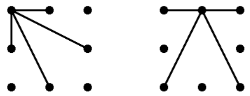
There are no tours with vertical symmetry (if we start at the extreme left-hand edge and proceed step by step along symmetric branches, sooner or later we must come to a second point on the axis of symmetry, at which point we have a closed loop from which the other two points in the axis of symmetry are missing). However, there are six tours with 180-degree rotational symmetry, and again they form three complementary pairs:



Other pairs of complementary tours exist, and indeed every tour can be shown to have a complementary tour. However, no tour can be converted into its complementary tour by reflection or 180-degree rotation. One tour or the other of a complementary pair must include a move between the two inside points, but the reflected or rotated tour will also include a move between these two points, and so cannot be the complementary tour.

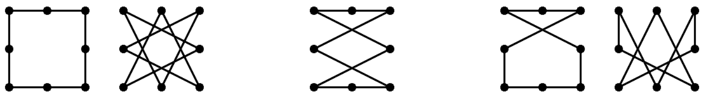
8-point networks (continued).

Another 8-point network can be created by using one-step orthogonal branches and “knight’s move” branches to connect the outside points of a 3 × 3 array, the central point not being used.

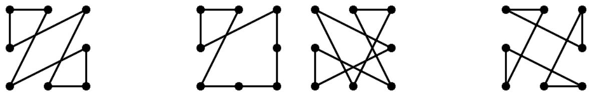


There being only two points on any diagonal, this network can host tours with diagonal symmetry as well as orthogonal and rotational symmetry.

There are in fact fifteen possible tours on this network, no fewer than nine having some kind of symmetry.



The tours on the left have every possible symmetry, and form a complementary pair. The central tour has horizontal, vertical, and 180-degree rotational symmetry, and is self-complementary by 90-degree rotation or by reflection in either diagonal. The tours on the right have horizontal symmetry, and form a complementary pair.

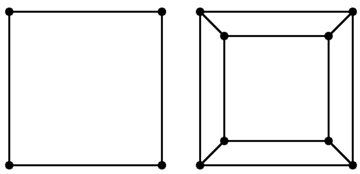
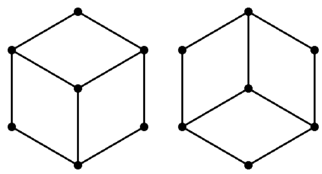


The tour on the left has symmetry about both diagonals and 180-degree rotational symmetry, and is self-complementary by horizontal or vertical reflection or by 90-degree rotation. The two central tours are symmetric about the NW-SE diagonal, and form a complementary pair. The tour on the right has 90-degree rotational symmetry, and is self-complementary by any orthogonal or diagonal reflection.

Networks on the vertices of regular convex solids

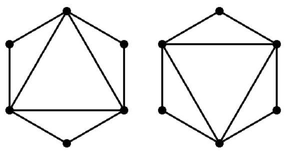
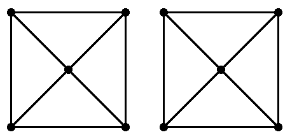
The vertices and edges of a convex solid with flat faces form a network, the vertices being the points of the network and the edges the branches. The most familiar of such solids is the cube, but with only three edges at each vertex this produces only a three-branch network, and so is not of interest to us here. However, its familiarity makes it a convenient example by which to illustrate the graphical representations that we shall use.

When illustrating a convex solid, we shall use a diagram to show the near-side faces, a diagram to show the far-side faces *as viewed from the inside* if they are arranged differently, and a diagram to show any faces edge-on to the viewer. This is shown for the cube in left-hand pair of the diagrams below. The near-faces diagram shows a single square face only. The far-faces-from-the-inside diagram would show the same, and so is not given separately. There remain the four faces edge-on to the viewer, which are shown in the second diagram. In these "edge-on" diagrams, the faces are tilted to make them visible, but triangles remain triangles, squares remain at least quadrilaterals, pentagons remain pentagons, and edge-to-edge and corner-to-corner connections are preserved.

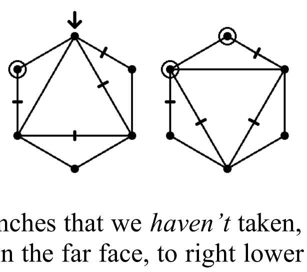
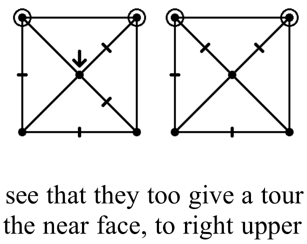



We shall also have occasion to look at a solid from in front of a vertex rather than a face. This is shown for the cube in the right-hand pair of diagrams. This time the far-side diagram is different from the near-side diagram, so it is presented separately, but there are no faces edge-on to the reader so there is no "edge-on" diagram.

All this applies to all convex solids with flat faces. A regular convex solid is one in which all the faces are equilateral, are equiangular, and have the same number of sides. There are five of them, the cube being the most familiar, but only the octahedron gives a four-branch network. This has eight triangular faces, four meeting at each vertex. The near-side and far-side-from-the-inside views as seen when looking at a face are shown in the left-hand pair of diagrams below (there are no faces edge-on to the viewer), and the views as seen when looking at a vertex in the right-hand pair (again there are no edge-on faces).

This network is easily toured. Suppose that we start at the top vertex on the near face (arrowed in the diagram below). On the near face, we can go to the right lower vertex, and then across to the left lower vertex. We now go to the far face, perhaps at the left upper vertex. On the far face, we go to the bottom vertex and then to the right upper vertex, and a move to the top brings us back home. The path we have taken is marked in the left-hand pair of diagrams below, points where we can make no further progress in the present diagram being ringed and the tour being picked up at the corresponding point in the other diagram, and moves between near and far sides appearing in both diagrams.

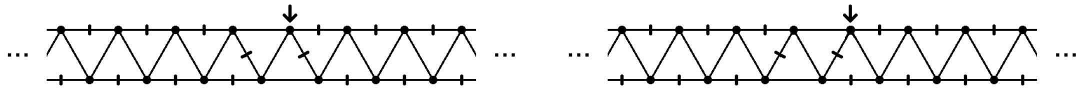



If we now look at the branches that we *haven't* taken, we see that they too give a tour (from top to left lower on the near face, to bottom on the far face, to right lower on the near face, to right upper and then left upper on the far face, and back to top), so we have a complementary pair. At first sight, these tours have nothing in common, but if we rotate the solid forward so that "top" becomes "near centre" and "bottom" becomes "far centre", our tour becomes "near centre to right lower to left lower to left upper to far centre to right upper to near centre" (see the pair of diagrams on the right), and we see that the unmarked complementary tour is simply our tour rotated through 180 degrees. So we have a tour which is self-complementary by rotation through 180 degrees about the line joining the top and bottom vertices.

Networks on the vertices of semi-regular convex solids

A semi-regular convex solid is one in which not all the faces have the same number of sides, but all of them are equilateral and equiangular, and the arrangement of faces around every vertex is the same. There are two infinite classes, the prisms and antiprisms, and thirteen others two of which have left-handed and right-handed forms. There are photographs and meticulous drawings in *Mathematical Models* by Cundy and Rollett, an excellent book remembered with affection from my schooldays, and doubtless in many other sources.

A prism is simply a pair of *n*-sided polygons placed one above the other with lines joining corresponding vertices. It has only three faces meeting at each vertex and so is not of interest to us here, but if we give the upper polygon a slight twist so that each vertex is above the mid-point of an edge of the lower polygon, and then join this vertex to the two vertices at the ends of the edge of the lower polygon, we obtain an “antiprism” which does have four faces meeting at each vertex (one of the polygons, and three triangles).



An antiprism of any size can be toured by zigzagging all the way round between the two polygons, but in that case the complementary moves are the moves along the edges of the polygons, and these form two separate half-tours and not one complete tour. For our purposes, it is better to start at a point on one polygon, to go right round this polygon until we are just about to join up, to move to one of the adjacent vertices of the other polygon (it doesn’t matter which), to go round this polygon in the opposite direction, and finally to return to the point from which we started. Thus in the left-hand diagram above, we might start on the upper polygon at the point immediately to the right of centre, move off right, go round the back (not shown in the diagram), come in again from the left, and move down and right to the lower polygon just before we would have joined up. We now go left round the lower polygon, and finally join up with our starting point.

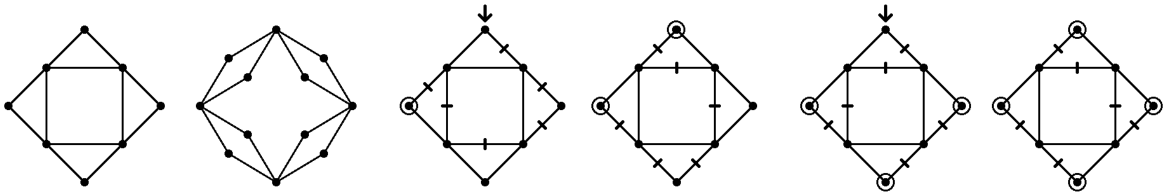
If we now look at the complementary moves, we see that they zigzag until they meet the point at which our tour crossed over, where they perform a Z before resuming their zigzag. So they too form a tour, and we have a complementary pair. Had we gone down and left when crossing over, we would have made the tour shown in the right-hand diagram, and the Z in the complementary tour would have been reflected.

There is more. An octahedron can be regarded as a triangular antiprism, and if we look at the tour we have shown here we see that it is exactly the tour that we made on the octahedron, the near and far triangles of the octahedron playing the roles of the upper and lower polygons here. And if we put a line through the centre of the antiprism and the mid-point of the Z, we see that both our tour and the complementary tour have 180-degree rotational symmetry about this line.

Networks on the vertices of semi-regular convex solids (continued).

For more general solids, the formal names can be very long and clumsy, and I shall refer to them by giving the numbers of sides of each polygon surrounding a vertex. So the cube (three squares) would be 4-4-4, the octahedron (four triangles) 3-3-3-3, and the antiprism with two *n*-sided polygons 3-3-3-*n*. The four of interest to us here will be the 3-4-3-4 ("cuboctahedron") with 12 vertices, the 3-4-4-4 ("rhombicuboctrahedron") with 18 vertices, the 3-5-3-5 ("icosidodecahedron") with 30 vertices, and the 3-4-5-4 ("rhombicosidodecahedron", which must be the longest word that I have had occasion to use in ordinary writing) with 60.[2]

Apart from the antiprisms, the simplest semi-regular solid with four faces at each vertex is the 3-4-3-4 (12 vertices, 8 triangular faces, 6 square faces, 24 edges). It is shown in the left-hand pair of diagrams below.

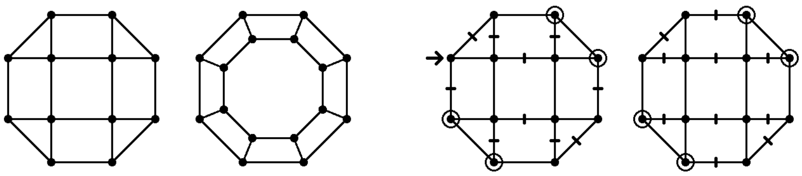


Each square face is surrounded by four triangular faces, and is opposite a square face similarly surrounded. The remaining four square faces, shown tilted in the second diagram, are parallel to the line of sight and have only corner-to-corner contact. The top, right, bottom, and left vertices are midway between the near and far sides, and are accessible from either.

The 3-4-3-4 admits several self-complementary tours. That shown in the middle pair of diagrams is self-complementary by reflection through the NW-SE diagonal plane. From midway top, we visit in turn near right upper, midway right, near right lower, near left lower, and near left upper. We have now visited all the corners of the near face, so we go to midway left and visit the corners of the far face similarly: far left lower, midway bottom, far right lower, far right upper, far left upper, and home.

The tour shown in the right-hand pair of diagrams is self-complementary by 180-degree rotation about the near face. From midway top, we visit near right upper, near left upper, and near left lower. The moves to near right lower and midway bottom are now barred to us because they will form part of the complementary tour, so we go to midway left and far left lower, and continue to midway bottom, near right lower (we have now visited all the corners of the near face), midway right, far right lower, far right upper, far left upper, and home.

There is also a tour self-complementary by 180-degree rotation about the line joining two opposing vertices.



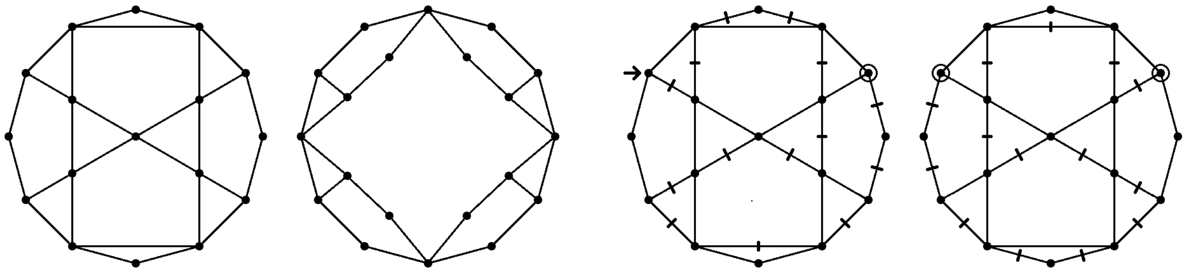
The 3-4-4-4 (24 vertices, 8 triangular faces, 18 square faces, 48 edges), shown in the left-hand pair of diagrams above, is built up from three loops of eight squares, one top to bottom and round, one left to right and round, and one edge-on to the viewer which is shown tilted in the second diagram. It admits the tour shown in the right-hand pair of diagrams, which is self-complementary by 90-degree rotation about the near face and has 180-degree rotational symmetry about this face. Here, the top and bottom halves of the solid are toured in turn. Starting at near left upper, we go to near top left and on to near top right, through to the far side, round to far right upper, and forward to near right upper. We have now visited all the vertices of the top half, so we move to near right lower, tour the bottom half similarly ending at near left lower, and a final move to near left upper completes the tour.

[2] A useful theorem, known to my school contemporary Douglas Draper as the "lost degrees" theorem, allows the number of vertices of a convex polyhedron to be calculated quickly and easily. Add up the corner angles of the polygons at each vertex, subtract the sum from 360, and we have the "lost degrees" at that vertex. The "lost degrees" theorem now states that the sum of these lost degrees over all the vertices of the solid is always 720. So the 3-4-3-4, whose faces have corner angles of 60 + 90 + 60 + 90 = 300, has 60 lost degrees at each vertex, and must have 12 vertices to bring the total to 720.

Draper was a skilled and enthusiastic modeller, both of complex polyhedra and of other mathematical phenomena. I still remember a splendid model of five cubes in a dodecahedron which used to decorate the maths room.

Networks on the vertices of semi-regular convex solids (continued).

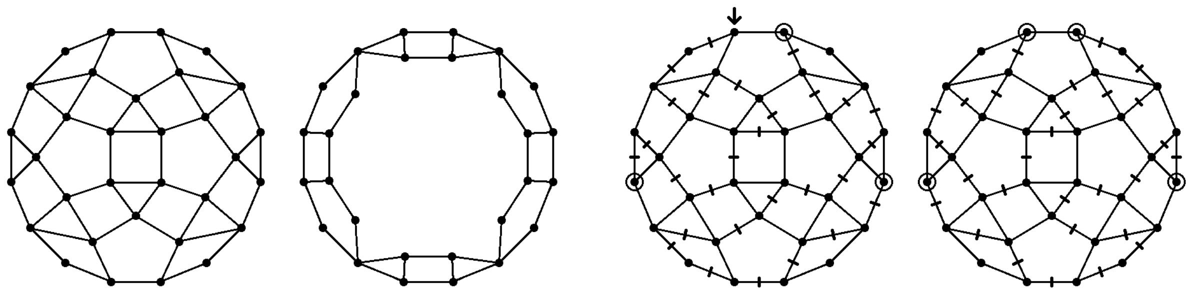
The 3-5-3-5 (30 vertices, 20 triangular faces, 12 pentagonal faces, 60 edges) is shown in the left-hand pair of diagrams below (here we are looking at a vertex). This and subsequent diagrams depart from exact perspective, but triangles remain triangles, squares remain at least quadrilaterals, and pentagons remain pentagons.



The 3-5-3-5 too is built up from three loops of eight polygons, but this time the polygons are pentagon-triangle-triangle-pentagon and so on round. Two adjacent pentagons have only corner-to-corner contact, as do two adjacent triangles, but pentagons adjacent to triangles have an edge in common. The three loops are arranged as before, one top to bottom and round, one left to right and round, and one edge-on to the viewer as shown tilted in the second diagram. The vertices at 12, 3, 6, and 9 o'clock, which are corner-to-corner meeting points of triangles or pentagons, are midway between the near and far sides and are accessible from either, and the paired vertices at 2, 4, 8, and 10 o'clock, which are the ends of edge-to-edge connections of triangles and pentagons, provide another route from the near to the far side and back.

This solid admits the tour shown in the right-hand pair of diagrams, which is self-complementary by 180-degree rotation about the central vertex. Starting at 10 o'clock on the near side, fourteen moves bring us to 2 o'clock having visited all the near-side vertices and the midway vertices at 12 and 3 o'clock. Move 15 takes us to 2 o'clock on the far side, and a further fourteen moves tour the far-side vertices plus the midway vertices at 6 and 9 o'clock. All this has taken 29 moves and we are now at 10 o'clock on the far side, and a final far-to-near move brings us back home. Looked at from the inside, as here, the far-side tour is essentially the near-side tour reflected left to right, but it has been altered to bypass the midway vertex at 12 o'clock and to visit that at 6 o'clock instead. If we go round and look at the far-side tour from the outside, it is exactly the same as the near-side tour except for including the midway vertex at 6 o'clock and bypassing that at 12 o'clock. Each of the vertices accessible from either side has been visited, but none has been used as a crossing point.

The complementary tour starts at 4 o'clock, and makes the near-to-far move at 8 o'clock.



The 3-4-5-4 (60 vertices, 20 triangular faces, 30 square faces, 12 pentagonal faces, 60 edges) is best viewed from in front of a square face. It is yet another solid based on three loops, though this time the loops are of twelve polygons, triangle-square-triangle-pentagon-square-pentagon and so on. The "edge-on" loop is shown tilted in the second diagram. Triangles and pentagons meet in points accessible from both sides, and squares meet triangle and pentagons in edges which provide further paths between the near and far sides.

The 3-4-5-4 admits the tour shown in the right-hand pair of diagrams, which is self-complementary by 180-degree rotation about the central square face. From near top left, twelve moves take us to near left lower. Move 13 takes us through to the far side, and a further thirteen moves take us to far right lower. Back to the near side, and sixteen moves take us to near top right; back to the far side, and fifteen moves take us to far top left; and a final far-to-near move brings us home. Again, each of the vertices accessible from either side has been visited, but none has been used as a crossing point. Seen from the inside, as diagrammed, the far-side tour is a left-to-right reflection of the near-side tour except that a central six-point lozenge (the central square and the two adjacent triangles) has been left unchanged; viewed from the outside, the tours are the same except that the lozenge has been reflected.

A class of indefinitely large networks

The only tours we have seen so far which have generalised to indefinitely large networks have been those on the antiprisms, and these have not been self-complementary. There is however a class of networks on which there are systematic self-complementary tours however large the network may be.

Consider a line of points of length $2n+1$, with branches connecting points n or $n+1$ points away in either direction. There are exactly two branches at every point on the line. At the centre, there is a branch to each end; at any other point, there are two branches towards the farther end but none towards the nearer end.

We can use these branches to tour the line. Suppose that we start at point 1, and always take the branch leading n points forwards or $n+1$ points backwards (one of these will always be possible); then we shall visit points $n+1$, $2n+1$, n, $2n$, $n-1$, $2n-1$, and so on in order, and after $2n$ of these moves we shall be at point $n+2$ having visited every other point on the way. Now a final move of length $n+1$ backwards will return us to our starting point.

There is a complementary tour whose moves are of length $n+1$ forwards or n backwards. From point 1, we visit the points in reverse order, $n+2$, 2, $n+3$, 3, and so on, and after $2n$ moves we are at point $n+1$ and a final backward move of length n will take us back home.

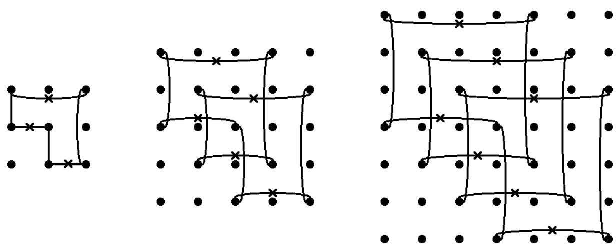
We shall make extensive use of sequences of $2n$ moves of this kind. To diagram every move is soon seen to be impracticable even if we show moves of length n on one side of the line and moves of length $n+1$ on the other side; it is not too clear even for a line of length 5, and is quite unreadable for lines of length 7 or beyond.

Instead, we shall diagram only the further move which would bring us back home, which we *don't* make, and we shall put a little cross on it to indicate that is it not made. So we shall diagram these sequences as follows.

We now consider a square array of side $2n+1$, the rows being numbered $1\ldots2n+1$ from the top and the columns $1\ldots2n+1$ from the left, with branches connecting points n or $n+1$ points away in either orthogonal direction. There are again four branches at every point. At the centre, there is a branch to each edge; at a non-central point on the central row, there are branches to the top and bottom edge, two branches towards the farther side edge, and none towards the nearer side edge; at a point on neither the central row nor the central column, there are two branches towards each of the farther edges but none towards the nearer edges.

Our method will be to start at point (1,1), to make $2n$ horizontal moves along row 1, then one vertical move, then $2n$ horizontal moves along the row we have now reached, then another vertical move, and so on until we have returned to our starting point.

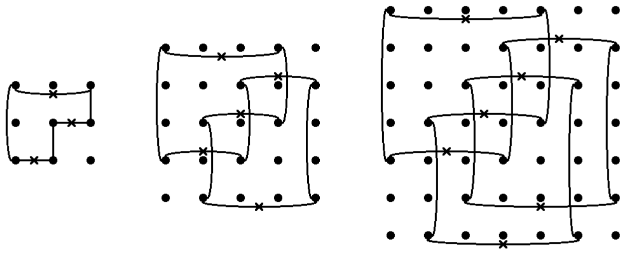
Let us first suppose that all our horizontal moves are rightwards of length n or leftwards of length $n+1$, and our vertical moves downwards of length $n+1$ or upwards of length n. This causes us to visit the rows in order 1, $n+2$, 2, $n+3$, and so on, and each vertical move brings us back to some point on the NW-SE diagonal. This gives us a pattern like one of those shown below.



If we now look at row 1 and at the rightmost column $2n+1$, we see the move from point $n+2$ to point 1 in row 1 is the one move that we *don't* make, and that the move from point $2n+1$ to point n in column $2n+1$ is the one move that we *do* make. The same can be shown to be true for every row i and column $2n+2-i$, and a consequence is that if we reflect the tour in the NE-SW diagonal we obtain the complementary tour.

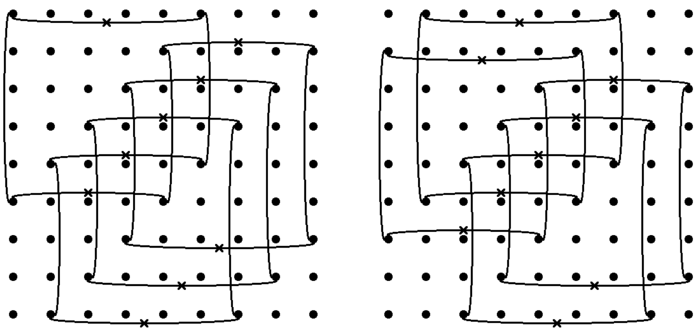
A class of indefinitely large networks (continued).

Alternatively, we can make our vertical moves of length *n* downwards or *n*+1 upwards. This gives patterns like those shown below.



If we now look at row 1 and column 1, we see that the move from point *n*+2 to point 1 in row 1 is again the move that we don't make, but now it is the move from point *n*+2 to point 1 in column 1 that we do make. The same can be shown to be true for every row *i* and column *i*, so this time if we reflect the tour in the NW-SE diagonal we obtain the complementary tour.

And there is more. The tour on the 5×5 array is also self-complementary by reflection in the NE-SW diagonal, though the other tours shown above are not. However, if we look at the tour on a 9×9 array (left-hand diagram below) and instead of starting from point (1,1) we start from point (1,2) as shown in the right-hand diagram, we see that the tour is again self-complementary by reflection in the NE-SW diagonal. It is also symmetric by 180-degree rotation.



All this generalizes, and we can say the following.

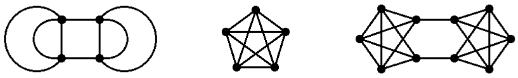
- Given a square array of side 2*n*+1, if we start at point (1,1), make 2*n* horizontal moves either of length *n* rightwards or of length *n*+1 leftwards, then a vertical move either of length *n*+1 downwards or of length *n* upwards, and repeat until we have made a total of $(2n+1)^2$ moves, we have a complete tour of the array which is self-complementary by reflection in the NW-SE diagonal.

- If instead we make vertical moves of length *n* downwards or of length *n*+1 upwards, we have a tour which is self-complementary by reflection in the NE-SW diagonal.

- If *n*=2*k* and we start from point (1,*k*) and do the same, we have a tour which is self-complementary by reflection in either diagonal and has 180-degree rotational symmetry.

Afterthought

After this paper had been put to bed, the thought occurred to me: do there exist tourable four-branch networks on which no pairs of complementary tours exist?



The answer was surely going to be Yes, and examples did not take too long to find. It can be done using only four points, as shown in the left-hand diagram above. From the left upper point, we can tour this by taking any of the upper-to-lower branches on the left, crossing to the right lower, taking any of the lower-to-upper branches on the right, and crossing back. This gives a range of possible tours, but each must contain the two left-to-right branches, so no two of them can be complementary.

If this use of more than one branch between the same two points is thought too artificial, it can easily be avoided. The middle diagram shows the five-point network in which every point is connected to every other point. If this is duplicated, and the two networks are turned on their sides and joined, we get the diagram on the right. We can tour this by starting at the inner left upper point, touring the left-hand network ending at the inner left lower (this can be done in six different ways), crossing to the inner right lower, touring the right-hand network ending at the inner right upper, and crossing back. This has given a whole host of tours, but each must contain the two left-to-right connecting branches, so no two can be complementary.

Acknowledgements

My main sources of information have been George Jelliss's Knight's Tour Notes, accessible at the time of writing on his "Mayhematics" site www.mayhematics.com, and David Singmaster's Sources in Recreational Mathematics, accessible at the time of writing on the Puzzle Museum site www.puzzlemuseum.com. That said, my days of visiting libraries are past and not everything is available on the Internet, and I have done no more than consult the sources conveniently available to me. I am well aware that most and probably all of what appears here has been previously reported, and if readers care to draw my attention to prior appearances I shall be very happy to insert references into any future edition of this document.