A note on a puzzle by Boothroyd and Conway

JDB reporting an observation by Megi Rychlíková in March 2023

Eureka 22 contains a report of the 1959 Problems Drive of the Archimedeans (a student mathematical society at Cambridge which I hope is still going strong). This was set by M. R. Boothroyd and J. H. Conway, and included the following:

> Five identical boxes originally contained the same quantity of sugar, but a little has been transferred from one to another. Given an uncalibrated balance and no weights, show how to identify both the heavy and the light box with the least number of weighings.

The solution was given as follows:

> Weigh (i) AB against CD, (ii) AC against BD, (iii) AD against BC. Then AB > CD, AC > BD, AD > BC implies A heavy and E light, and similarly for all other combinations with no balancing at any stage. If, say, AB = CD, then E must be normal, and neither of the other two weighings can balance. Then if AC > BD, AD > BC, A must be heavy and B light. All possibilities are of these two types.

I recently threw this problem at my daughter Megi, and she came up with a simpler solution:

> Weigh (i) AB against CD, (ii) A against B, (iii) C against D.

To my surprise, this also worked.

It then occurred to me to wonder: if we weigh A against B and C against D first, can we find a third weighing with just one box on each side which will resolve the situation?

The answer turned out to be Yes. Suppose A > B and C > D, then either B has given to C or D has given to A, and weighing A against C (or B against D, or indeed any weighing which we have not yet done) will tell us which is the case. Alternatively, suppose A > B and C = D; then either B has given to A, when E will be normal, or E has given to A, when E will be light, or B has given to E, when E will be heavy, and weighing E against either C or D will tell us which.

So, if we are allowed not to decide on the third weighing until we have seen the results of the first two, we can solve the problem without at any time putting more than more than one box on either side of the balance.

There is nothing in *Eureka* 22 to suggest that this alternative solution was noticed at the time, nor does there appear to be any mention of the matter in the next issue of *Eureka*. I do not know if attention has been drawn to it since.